

LONG-TIME STABILITY OF MULTI-DIMENSIONAL NONCHARACTERISTIC VISCOUS BOUNDARY LAYERS

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ABSTRACT. We establish long-time stability of multi-dimensional noncharacteristic boundary layers of a class of hyperbolic-parabolic systems including the compressible Navier-Stokes equations with inflow [outflow] boundary conditions, under the assumption of strong spectral, or uniform Evans, stability. Evans stability has been verified for small-amplitude layers by Guès, Métivier, Williams, and Zumbrun. For large-amplitude layers, it may be efficiently checked numerically, as done in the one-dimensional case by Costanzino, Humpherys, Nguyen, and Zumbrun.

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1. INTRODUCTION

We consider a boundary layer, or stationary solution,

$$(1.1) \quad \tilde{U} = \bar{U}(x_1), \quad \lim_{z \rightarrow +\infty} \bar{U}(z) = U_+, \quad \bar{U}(0) = \bar{U}_0$$

of a system of conservation laws on the quarter-space

$$(1.2) \quad \tilde{U}_t + \sum_j F^j(\tilde{U})_{x_j} = \sum_{jk} (B^{jk}(\tilde{U})\tilde{U}_{x_k})_{x_j}, \quad x \in \mathbb{R}_+^d = \{x_1 > 0\}, \quad t > 0,$$

$\tilde{U}, F^j \in \mathbb{R}^n$, $B^{jk} \in \mathbb{R}^{n \times n}$, with initial data $\tilde{U}(x, 0) = \tilde{U}_0(x)$ and Dirichlet type boundary conditions specified in (1.5), (1.6) below. A fundamental question connected to the physical motivations from aerodynamics is whether or not such boundary layer solutions are *stable* in the sense of PDE, i.e., whether or not a sufficiently small perturbation of \bar{U} remains close to \bar{U} , or converges time-asymptotically to \bar{U} , under the evolution of (1.2). That is the question we address here.

1.1. Equations and assumptions. We consider the general hyperbolic-parabolic system of conservation laws (1.2) in conserved variable \tilde{U} , with

$$\tilde{U} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_1^{jk} & b_2^{jk} \end{pmatrix},$$

$\tilde{u} \in \mathbb{R}^{n-r}$, and $\tilde{v} \in \mathbb{R}^r$, where

$$\Re \sigma \sum_{jk} b_2^{jk} \xi_j \xi_k \geq \theta |\xi|^2 > 0, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Following [MaZ4, Z3, Z4], we assume that equations (1.2) can be written, alternatively, after a triangular change of coordinates

$$(1.3) \quad \tilde{W} := \tilde{W}(\tilde{U}) = \begin{pmatrix} \tilde{w}^I(\tilde{u}) \\ \tilde{w}^{II}(\tilde{u}, \tilde{v}) \end{pmatrix},$$

in the *quasilinear, partially symmetric hyperbolic-parabolic form*

$$(1.4) \quad \tilde{A}^0 \tilde{W}_t + \sum_j \tilde{A}^j \tilde{W}_{x_j} = \sum_{jk} (\tilde{B}^{jk} \tilde{W}_{x_k})_{x_j} + \tilde{G},$$

where, defining $\tilde{W}_+ := \tilde{W}(U_+)$,

(A1) $\tilde{A}^j(\tilde{W}_+)$, \tilde{A}^0 , \tilde{A}_{11}^1 are symmetric, \tilde{A}^0 block diagonal, $\tilde{A}^0 \geq \theta_0 > 0$,

(A2) for each $\xi \in \mathbb{R}^d \setminus \{0\}$, no eigenvector of $\sum_j \xi_j \tilde{A}^j (\tilde{A}^0)^{-1} (\tilde{W}_+)$ lies in the kernel of $\sum_{jk} \xi_j \xi_k \tilde{B}^{jk} (\tilde{A}^0)^{-1} (\tilde{W}_+)$,

(A3) $\tilde{B}^{jk} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b}^{jk} \end{pmatrix}$, $\sum \tilde{b}^{jk} \xi_j \xi_k \geq \theta |\xi|^2$, and $\tilde{G} = \begin{pmatrix} 0 \\ \tilde{g} \end{pmatrix}$ with $\tilde{g}(\tilde{W}_x, \tilde{W}_x) = \mathcal{O}(|\tilde{W}_x|^2)$.

Along with the above structural assumptions, we make the following technical hypotheses:

(H0) $F^j, B^{jk}, \tilde{A}^0, \tilde{A}^j, \tilde{B}^{jk}, \tilde{W}(\cdot), \tilde{g}(\cdot, \cdot) \in C^s$, with $s \geq [(d-1)/2] + 5$ in our analysis of linearized stability, and $s \geq s(d) := [(d-1)/2] + 7$ in our analysis of nonlinear stability.

(H1) \tilde{A}_1^{11} is either strictly positive or strictly negative, that is, either $\tilde{A}_1^{11} \geq \theta_1 > 0$, or $\tilde{A}_1^{11} \leq -\theta_1 < 0$. (We shall call these cases the *inflow case* or *outflow case*, correspondingly.)

(H2) The eigenvalues of $dF^1(U_+)$ are distinct and nonzero.

(H3) The eigenvalues of $\sum_j dF_{\pm}^j \xi_j$ have constant multiplicity with respect to $\xi \in \mathbb{R}^d$, $\xi \neq 0$.

(H4) The set of branch points of the eigenvalues of $(\tilde{A}^1)^{-1}(i\tau\tilde{A}^0 + \sum_{j \neq 1} i\xi_j \tilde{A}^j)_{\pm}$, $\tau \in \mathbb{R}$, $\tilde{\xi} \in \mathbb{R}^{d-1}$ is the (possibly intersecting) union of finitely many smooth curves $\tau = \eta_q^{\pm}(\tilde{\xi})$, on which the branching eigenvalue has constant multiplicity s_q (by definition ≥ 2).

Condition (H1) corresponds to hyperbolic–parabolic noncharacteristicity, while (H2) is the condition for the hyperbolicity at U_+ of the associated first-order hyperbolic system obtained by dropping second-order terms. The assumptions (A1)–(A3) and (H0)–(H2) are satisfied for gas dynamics and MHD with van der Waals equation of state under inflow or outflow conditions; see discussions in [MaZ4, CHNZ, GMWZ5, GMWZ6]. Condition (H3) holds always for gas dynamics, but fails always for MHD in dimension $d \geq 2$. Condition (H4) is a technical requirement of the analysis introduced in [Z2]. It is satisfied always in dimension $d = 2$ or for rotationally invariant systems in dimensions $d \geq 2$, for which it serves only to define notation; in particular, it holds always for gas dynamics.

We also assume:

(B) Dirichlet boundary conditions in \tilde{W} -coordinates:

$$(1.5) \quad (\tilde{w}^I, \tilde{w}^{II})(0, \tilde{x}, t) = \tilde{h}(\tilde{x}, t) := (\tilde{h}_1, \tilde{h}_2)(\tilde{x}, t)$$

for the inflow case, and

$$(1.6) \quad \tilde{w}^{II}(0, \tilde{x}, t) = \tilde{h}(\tilde{x}, t)$$

for the outflow case, with $x = (x_1, \tilde{x}) \in \mathbb{R}^d$.

This is sufficient for the main physical applications; the situation of more general, Neumann and mixed-type boundary conditions on the parabolic variable v can be treated as discussed in [GMWZ5, GMWZ6].

Example 1.1. The main example we have in mind consists of *laminar solutions* $(\rho, u, e)(x_1, t)$ of the compressible Navier–Stokes equations

$$(1.7) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u^t u) + \nabla p = \varepsilon \mu \Delta u + \varepsilon(\mu + \eta) \nabla \operatorname{div} u \\ \partial_t(\rho E) + \operatorname{div}((\rho E + p)u) = \varepsilon \kappa \Delta T + \varepsilon \mu \operatorname{div}((u \cdot \nabla)u) \\ \quad + \varepsilon(\mu + \eta) \nabla(u \cdot \operatorname{div} u), \end{cases}$$

$x \in \mathbb{R}^d$, on a half-space $x_1 > 0$, where ρ denotes density, $u \in \mathbb{R}^d$ velocity, e specific internal energy, $E = e + \frac{|u|^2}{2}$ specific total energy, $p = p(\rho, e)$ pressure, $T = T(\rho, e)$ temperature, $\mu > 0$ and $|\eta| \leq \mu$ first and second coefficients of viscosity, $\kappa > 0$ the coefficient of heat

conduction, and $\varepsilon > 0$ (typically small) the reciprocal of the Reynolds number, with no-slip *suction-type* boundary conditions on the velocity,

$$u_j(0, x_2, \dots, x_d) = 0, \quad j \neq 1 \quad \text{and} \quad u_1(0, x_2, \dots, x_d) = V(x) < 0,$$

and prescribed temperature, $T(0, x_2, \dots, x_d) = T_{wall}(\tilde{x})$. Under the standard assumptions $p_\rho, T_e > 0$, this can be seen to satisfy all of the hypotheses (A1)–(A3), (H0)–(H4), (B) in the *outflow case* (1.6); indeed these are satisfied also under much weaker van der Waals gas assumptions [MaZ4, Z3, CHNZ, GMWZ5, GMWZ6]. In particular, boundary-layer solutions are of noncharacteristic type, scaling as $(\rho, u, e) = (\bar{\rho}, \bar{u}, \bar{e})(x_1/\varepsilon)$, with layer thickness $\sim \varepsilon$ as compared to the $\sim \sqrt{\varepsilon}$ thickness of the characteristic type found for an impermeable boundary.

This corresponds to the situation of an airfoil with microscopic holes through which gas is pumped from the surrounding flow, the microscopic suction imposing a fixed normal velocity while the macroscopic surface imposes standard temperature conditions as in flow past a (nonporous) plate. This configuration was suggested by Prandtl and tested experimentally by G.I. Taylor as a means to reduce drag by stabilizing laminar flow; see [S, Bra]. It was implemented in the NASA F-16XL experimental aircraft program in the 1990's with reported 25% reduction in drag at supersonic speeds [Bra].¹ Possible mechanisms for this reduction are smaller thickness $\sim \varepsilon \ll \sqrt{\varepsilon}$ of noncharacteristic boundary layers as compared to characteristic type, and greater stability, delaying the transition from laminar to turbulent flow. In particular, stability properties appear to be quite important for the understanding of this phenomenon. For further discussion, including the related issues of matched asymptotic expansion, multi-dimensional effects, and more general boundary configurations, see [GMWZ5].

Example 1.2. Alternatively, we may consider the compressible Navier–Stokes equations (1.7) with *blowing-type* boundary conditions

$$u_j(0, x_2, \dots, x_d) = 0, \quad j \neq 1 \quad \text{and} \quad u_1(0, x_2, \dots, x_d) = V(x) > 0,$$

and prescribed temperature and pressure

$$T(0, x_2, \dots, x_d) = T_{wall}(\tilde{x}), \quad p(0, x_2, \dots, x_d) = p_{wall}(\tilde{x})$$

(equivalently, prescribed temperature and density). Under the standard assumptions $p_\rho, T_e > 0$ on the equation of state (alternatively, van der Waals gas assumptions), this can be seen to satisfy hypotheses (A1)–(A3), (H0)–(H4), (B) in the *inflow case* (1.5).

Lemma 1.3 ([MaZ3, Z3, GMWZ5, NZ]). *Given (A1)–(A3) and (H0)–(H2), a standing wave solution (1.1) of (1.2), (B) satisfies*

$$(1.8) \quad \left| (d/dx_1)^k (\bar{U} - U_+) \right| \leq C e^{-\theta x_1}, \quad 0 \leq k \leq s+1,$$

as $x_1 \rightarrow +\infty$, s as in (H0). Moreover, a solution, if it exists, is in the inflow or strictly parabolic case unique; in the outflow case it is locally unique.

Proof. See Lemma 1.3, [NZ]. □

¹ See also NASA site <http://www.dfrc.nasa.gov/Gallery/photo/F-16XL2/index.html>

1.2. The Evans condition and strong spectral stability. The linearized equations of (1.2), (B) about \bar{U} are

$$(1.9) \quad U_t = LU := \sum_{j,k} (B^{jk} U_{x_k})_{x_j} - \sum_j (A^j U)_{x_j}$$

with initial data $U(0) = U_0$ and boundary conditions in (linearized) \tilde{W} -coordinates of

$$W(0, \tilde{x}, t) := (w^I, w^{II})^T(0, \tilde{x}, t) = h$$

for the inflow case, and

$$w^{II}(0, \tilde{x}, t) = h$$

for the outflow case, with $x = (x_1, \tilde{x}) \in \mathbb{R}^d$, where $W := (\partial \tilde{W} / \partial U)(\bar{U})U$.

A necessary condition for linearized stability is weak spectral stability, defined as nonexistence of unstable spectra $\Re \lambda > 0$ of the linearized operator L about the wave. As described in Section 2.1.1, this is equivalent to nonvanishing for all $\tilde{\xi} \in \mathbb{R}^{d-1}$, $\Re \lambda > 0$ of the *Evans function*

$$D_L(\tilde{\xi}, \lambda)$$

(defined in (2.8)), a Wronskian associated with the Fourier-transformed eigenvalue ODE.

Definition 1.4. We define *strong spectral stability* as *uniform Evans stability*:

$$(D) \quad |D_L(\tilde{\xi}, \lambda)| \geq \theta(C) > 0$$

for $(\tilde{\xi}, \lambda)$ on bounded subsets $C \subset \{\tilde{\xi} \in \mathbb{R}^{d-1}, \Re \lambda \geq 0\} \setminus \{0\}$.

For the class of equations we consider, this is equivalent to the uniform Evans condition of [GMWZ5, GMWZ6], which includes an additional high-frequency condition that for these equations is always satisfied (see Proposition 3.8, [GMWZ5]). A fundamental result proved in [GMWZ5] is that small-amplitude noncharacteristic boundary-layers are always strongly spectrally stable.²

Proposition 1.5 ([GMWZ5]). *Assuming (A1)-(A3), (H0)-(H3), (B) for some fixed end-state (or compact set of endstates) U_+ , boundary layers with amplitude*

$$\|\bar{U} - U_+\|_{L^\infty[0, +\infty]}$$

sufficiently small satisfy the strong spectral stability condition (D).

As demonstrated in [SZ], stability of large-amplitude boundary layers may fail for the class of equations considered here, even in a single space dimension, so there is no such general theorem in the large-amplitude case. Stability of large-amplitude boundary-layers may be checked efficiently by numerical Evans computations as in [BDG, Br1, Br2, BrZ, HuZ, BHRZ, HLZ, CHNZ, HLYZ1, HLYZ2].

² The result of [GMWZ5] applies also to more general types of boundary conditions and in some situations to systems with variable multiplicity characteristics, including, in some parameter ranges, MHD.

1.3. Main results. Our main results are as follows.

Theorem 1.6 (Linearized stability). *Assuming (A1)-(A3), (H0)-(H4), (B), and strong spectral stability (D), we obtain asymptotic $L^1 \cap H^{[(d-1)/2]+5} \rightarrow L^p$ stability of (1.9) in dimension $d \geq 2$, and any $2 \leq p \leq \infty$, with rate of decay*

$$(1.10) \quad \begin{aligned} |U(t)|_{L^2} &\leq C(1+t)^{-\frac{d-1}{4}}(|U_0|_{L^1 \cap H^3} + E_0), \\ |U(t)|_{L^p} &\leq C(1+t)^{-\frac{d}{2}(1-1/p)+1/2p}(|U_0|_{L^1 \cap H^{[(d-1)/2]+5}} + E_0), \end{aligned}$$

provided that the initial perturbations U_0 are in $L^1 \cap H^3$ for $p = 2$, or in $L^1 \cap H^{[(d-1)/2]+5}$ for $p > 2$, and boundary perturbations h satisfy

$$(1.11) \quad \begin{aligned} |h(t)|_{L^2_{\tilde{x}}} &\leq E_0(1+t)^{-(d+1)/4}, \\ |h(t)|_{L^\infty_{\tilde{x}}} &\leq E_0(1+t)^{-d/2} \\ |\mathcal{D}_h(t)|_{L^1_{\tilde{x}} \cap H^{[(d-1)/2]+5}_{\tilde{x}}} &\leq E_0(1+t)^{-d/2-\epsilon}, \end{aligned}$$

where $\mathcal{D}_h(t) := |h_t| + |h_{\tilde{x}}| + |h_{\tilde{x}\tilde{x}}|$, E_0 is some positive constant, and $\epsilon > 0$ is arbitrary small for the case $d = 2$ and $\epsilon = 0$ for $d \geq 3$.

Theorem 1.7 (Nonlinear stability). *Assuming (A1)-(A3), (H0)-(H4), (B), and strong spectral stability (D), we obtain asymptotic $L^1 \cap H^s \rightarrow L^p \cap H^s$ stability of \bar{U} as a solution of (1.2) in dimension $d \geq 2$, for $s \geq s(d)$ as defined in (H0), and any $2 \leq p \leq \infty$, with rate of decay*

$$(1.12) \quad \begin{aligned} |\tilde{U}(t) - \bar{U}|_{L^p} &\leq C(1+t)^{-\frac{d}{2}(1-1/p)+1/2p}(|U_0|_{L^1 \cap H^s} + E_0) \\ |\tilde{U}(t) - \bar{U}|_{H^s} &\leq C(1+t)^{-\frac{d-1}{4}}(|U_0|_{L^1 \cap H^s} + E_0), \end{aligned}$$

provided that the initial perturbations $U_0 := \tilde{U}_0 - \bar{U}$ are sufficiently small in $L^1 \cap H^s$ and boundary perturbations $h(t) := \tilde{h}(t) - W(\bar{U}_0)$ satisfy (1.11) and

$$(1.13) \quad \mathcal{B}_h(t) \leq E_0(1+t)^{-\frac{d-1}{4}},$$

with sufficiently small E_0 , where the boundary measure \mathcal{B}_h is defined as

$$(1.14) \quad \mathcal{B}_h(t) := |h|_{H^s(\tilde{x})} + \sum_{i=0}^{[(s+1)/2]} |\partial_t^i h|_{L^2(\tilde{x})}$$

for the outflow case, and similarly

$$(1.15) \quad \mathcal{B}_h(t) := |h|_{H^s(\tilde{x})} + \sum_{i=0}^{[(s+1)/2]} |\partial_t^i h_2|_{L^2(\tilde{x})} + \sum_{i=0}^s |\partial_t^i h_1|_{L^2(\tilde{x})}$$

for the inflow case.

Combining Theorem 1.7 and Proposition 1.5, we obtain the following small-amplitude stability result, applying in particular to the motivating situation of Example 1.1.

Corollary 1.8. *Assuming (A1)-(A3), (H0)-(H4), (B) for some fixed endstate (or compact set of endstates) U_+ , boundary layers with amplitude*

$$\|\bar{U} - U_+\|_{L^\infty[0,+\infty]}$$

sufficiently small are linearly and nonlinearly stable in the sense of Theorems 1.6 and 1.7.

Remark 1.9. The obtained rate of decay in L^2 may be recognized as that of a $(d-1)$ -dimensional heat kernel, and the obtained rate of decay in L^∞ as that of a d -dimensional heat kernel. We believe that the sharp rate of decay in L^2 is rather that of a d -dimensional heat kernel and the sharp rate of decay in L^∞ dependent on the characteristic structure of the associated inviscid equations, as in the constant-coefficient case [HoZ1, HoZ2].

Remark 1.10. In one dimension, strong spectral stability is necessary for linearized asymptotic stability; see Theorem 1.6, [NZ]. However, in multi-dimensions, it appears likely that, as in the shock case [Z3], there are intermediate possibilities between strong and weak spectral stability for which linearized stability might hold with degraded rates of decay. In any case, the gap between the necessary weak spectral and the sufficient strong spectral stability conditions concerns only pure imaginary spectra $\Re\lambda = 0$ on the boundary between strictly stable and unstable half-planes, so this should not interfere with investigation of physical stability regions.

1.4. Discussion and open problems. Asymptotic stability, without rates of decay, has been shown for small amplitude noncharacteristic “normal” boundary layers of the isentropic compressible Navier–Stokes equations with outflow boundary conditions and vanishing transverse velocity in [KK], using energy estimates. Corollary 1.8 recovers this existing result and extends it to the general arbitrary transverse velocity, outflow or inflow, and isentropic or nonisentropic (full compressible Navier–Stokes) case, in addition giving asymptotic rates of decay. Moreover, we treat perturbations of boundary as well as initial data, as previous time-asymptotic investigations (with the exception of direct predecessors [YZ, NZ]) do not. As discussed in Appendix A, the type of boundary layer relevant to the drag-reduction strategy discussed in Examples 1.1–1.2 is a noncharacteristic “transverse” type with constant normal velocity, complementary to the normal type considered in [KK].

The large-amplitude asymptotic stability result of Theorem 1.7 extends to multi dimensions corresponding one-dimensional results of [YZ, NZ], reducing the problem of stability to verification of a numerically checkable Evans condition. See also the related, but technically rather different, work on the small viscosity limit in [MZ, GMWZ5, GMWZ6]. By a combination of numerical Evans function computations and asymptotic ODE estimates, spectral stability has been checked for *arbitrary amplitude* noncharacteristic boundary layers of the one-dimensional isentropic compressible Navier–Stokes equations in [CHNZ]. Extensions to the nonisentropic and multi-dimensional case should be possible by the methods used in [HLyZ1] and [HLyZ2] respectively to treat the related shock stability problem.

This (investigation of large-amplitude spectral stability) would be a very interesting direction for further investigation. In particular, note that it is large-amplitude stability that is relevant to drag-reduction at flight speeds, since the transverse relative velocity (i.e., velocity parallel to the airfoil) is zero at the wing surface and flight speed outside a thin boundary layer, so that variation across the boundary layer is substantial. We discuss this problem further in Appendix A for the model isentropic case.

Our method of analysis follows the basic approach introduced in [Z2, Z3, Z4] for the study of multi-dimensional shock stability and we are able to make use of much of that analysis without modification. However, there are some new difficulties to be overcome in the boundary-layer case.

The main new difficulty is that the boundary-layer case is analogous to the *undercompressive shock* case rather than the more favorable *Lax shock* case emphasized in [Z3], in that $G_{y_1} \not\sim t^{-1/2}G$ as in the Lax shock case but rather $G_{y_1} \sim (e^{-\theta|y_1|} + t^{-1/2})G$, $\theta > 0$, as in the undercompressive case. This is a significant difficulty; indeed, for this reason, the undercompressive shock analysis was carried out in [Z3] only in nonphysical dimensions $d \geq 4$. On the other hand, there is no translational invariance in the boundary layer problem, so no zero-eigenvalue and no pole of the resolvent kernel at the origin for the one-dimensional operator, and in this sense G is somewhat better in the boundary layer than in the shock case.

Thus, the difficulty of the present problem is roughly intermediate to that of the Lax and undercompressive shock cases. Though the undercompressive shock case is still open in multi-dimensions for $d \leq 3$, the slight advantage afforded by lack of pole terms allows us to close the argument in the boundary-layer case. Specifically, thanks to the absence of pole terms, we are able to get a slightly improved rate of decay in $L^\infty(x_1)$ norms, though our $L^2(x_1)$ estimates remain the same as in the shock case. By keeping track of these improved sup norm bounds throughout the proof, we are able to close the argument without using detailed pointwise bounds as in the one-dimensional analyses of [HZ, RZ].

Other difficulties include the appearance of boundary terms in integrations by parts, which makes the auxiliary energy estimates by which we control high-frequency effects considerably more difficult in the boundary-layer than in the shock-layer case, and the treatment of boundary perturbations. In terms of the homogeneous Green function G , boundary perturbations lead by a standard duality argument to contributions consisting of integrals on the boundary of perturbations against various derivatives of G , and these are a bit too singular as time goes to zero to be absolutely integrable. Following the strategy introduced in [YZ, NZ], we instead use duality to convert these to less singular integrals over the whole space, that *are* absolutely integrable in time. However, we make a key improvement here over the treatment in [YZ, NZ], integrating against an exponentially decaying test function to obtain terms of exactly the same form already treated for the homogeneous problem. This is necessary for us in the multi-dimensional case, for which we have insufficient information about individual parts of the solution operator to estimate them separately as in [YZ, NZ], but makes things much more transparent also in the one-dimensional case.

Among physical systems, our hypotheses appear to apply to and essentially only to the case of compressible Navier–Stokes equations with inflow or outflow boundary conditions. However, the method of analysis should apply, with suitable modifications, to more general situations such as MHD; see for example the recent results on the related small-viscosity problem in [GMWZ5, GMWZ6]. The extension to MHD is a very interesting open problem.

Finally, as pointed out in Remark 1.10, the strong spectral stability condition does not appear to be necessary for asymptotic stability. It would be interesting to develop a refined stability condition similarly as was done in [SZ, Z2, Z3, Z4] for the shock case.

2. RESOLVENT KERNEL: CONSTRUCTION AND LOW-FREQUENCY BOUNDS

In this section, we briefly recall the construction of resolvent kernel and then establish the pointwise low-frequency bounds on $G_{\tilde{\xi},\lambda}$, by appropriately modifying the proof in [Z3] in the boundary layer context [YZ, NZ].

2.1. Construction. We construct a representation for the family of elliptic Green distributions $G_{\tilde{\xi},\lambda}(x_1, y_1)$,

$$(2.1) \quad G_{\tilde{\xi},\lambda}(\cdot, y_1) := (L_{\tilde{\xi}} - \lambda)^{-1} \delta_{y_1}(\cdot),$$

associated with the ordinary differential operators $(L_{\tilde{\xi}} - \lambda)$, i.e. the resolvent kernel of the Fourier transform $L_{\tilde{\xi}}$ of the linearized operator L of (1.9). To do so, we study the homogeneous eigenvalue equation $(L_{\tilde{\xi}} - \lambda)U = 0$, or

$$(2.2) \quad \overbrace{(B^{11}U')' - (A^1U)'}^{L_0U} - i \sum_{j \neq 1} A^j \xi_j U + i \sum_{j \neq 1} B^{j1} \xi_j U' + i \sum_{k \neq 1} (B^{1k} \xi_k U)' - \sum_{j,k \neq 1} B^{jk} \xi_j \xi_k U - \lambda U = 0,$$

with boundary conditions (translated from those in W -coordinates)

$$(2.3) \quad \begin{pmatrix} A_{11}^1 - A_{12}^1 (b_2^{11})^{-1} b_1^{11} & 0 \\ b_1^{11} & b_2^{11} \end{pmatrix} U(0) \equiv \begin{pmatrix} * \\ 0 \end{pmatrix}$$

where $*$ = 0 for the inflow case and is arbitrary for the outflow case.

Define

$$\Lambda^{\tilde{\xi}} := \bigcap_{j=1}^n \Lambda_j^+(\tilde{\xi})$$

where $\Lambda_j^+(\tilde{\xi})$ denote the open sets bounded on the left by the algebraic curves $\lambda_j^+(\xi_1, \tilde{\xi})$ determined by the eigenvalues of the symbols $-\xi^2 B_+ - i\xi A_+$ of the limiting constant-coefficient operators

$$L_{\tilde{\xi}+} w := B_+ w'' - A_+ w'$$

as ξ_1 is varied along the real axis, with $\tilde{\xi}$ held fixed. The curves $\lambda_j^+(\cdot, \tilde{\xi})$ comprise the essential spectrum of operators $L_{\tilde{\xi}+}$. Let Λ denote the set of $(\tilde{\xi}, \lambda)$ such that $\lambda \in \Lambda^{\tilde{\xi}}$.

For $(\tilde{\xi}, \lambda) \in \Lambda^{\tilde{\xi}}$, introduce locally analytically chosen (in $\tilde{\xi}, \lambda$) matrices

$$(2.4) \quad \Phi^+ = (\phi_1^+, \dots, \phi_k^+), \quad \Phi^0 = (\phi_{k+1}^0, \dots, \phi_{n+r}^0),$$

and

$$(2.5) \quad \Phi = (\Phi^+, \Phi^0),$$

whose columns span the subspaces of solutions of (2.2) that, respectively, decay at $x = +\infty$ and satisfy the prescribed boundary conditions at $x = 0$, and locally analytically chosen matrices

$$(2.6) \quad \Psi^0 = (\psi_1^0, \dots, \psi_k^0), \quad \Psi^+ = (\psi_{k+1}^+, \dots, \psi_{n+r}^+)$$

and

$$(2.7) \quad \Psi = (\Psi^0, \Psi^+).$$

whose columns span complementary subspaces. The existence of such matrices is guaranteed by the general Evans function framework of [AGJ, GZ, MaZ3]; see in particular [Z3, NZ]. That dimensions sum to $n + r$ follows by a general result of [GMWZ5]; see also [SZ].

2.1.1. *The Evans function.* Following [AGJ, GZ, SZ], we define on Λ the *Evans function*

$$(2.8) \quad D_L(\tilde{\xi}, \lambda) := \det(\Phi^0, \Phi^+)_{|x=0}.$$

Evidently, eigenfunctions decaying at $+\infty$ and satisfying the prescribed boundary conditions at $x_1 = 0$ occur precisely when the subspaces $\text{span } \Phi^0$ and $\text{span } \Phi^+$ intersect, i.e., at zeros of the Evans function

$$D_L(\tilde{\xi}, \lambda) = 0.$$

The Evans function as constructed here is locally analytic in $(\tilde{\xi}, \lambda)$, which is all that we need for our analysis; we prescribe different versions of the Evans function as needed on different neighborhoods of Λ . Note that Λ includes all of $\{\tilde{\xi} \in \mathbb{R}^{d-1}, \Re \lambda \geq 0\} \setminus \{0\}$, so that Definition 1.4 is well-defined and equivalent to simple nonvanishing, away from the origin $(\tilde{\xi}, \lambda) = (0, 0)$. To make sense of this definition near the origin, we must insist that the matrices Φ^j in (2.8) remain *uniformly bounded*, a condition that can always be achieved by limiting the neighborhood of definition.

For the class of equations we consider, the Evans function may in fact be extended continuously along rays through the origin [R2, MZ, GMWZ5, GMWZ6].

2.1.2. *Basic representation formulae.* Define the solution operator from y_1 to x_1 of ODE $(L_{\tilde{\xi}} - \lambda)U = 0$, denoted by $\mathcal{F}^{y_1 \rightarrow x_1}$, as

$$\mathcal{F}^{y_1 \rightarrow x_1} = \Phi(x_1, \lambda) \Phi^{-1}(y_1, \lambda)$$

and the projections $\Pi_{y_1}^0, \Pi_{y_1}^+$ on the stable manifolds at $0, +\infty$ as

$$\Pi_{y_1}^+ = \begin{pmatrix} \Phi^+(y_1) & 0 \end{pmatrix} \Phi^{-1}(y_1), \quad \Pi_{y_1}^0 = \begin{pmatrix} 0 & \Phi^0(y_1) \end{pmatrix} \Phi^{-1}(y_1).$$

We define also the dual subspaces of solutions of $(L_{\tilde{\xi}}^* - \lambda^*)\tilde{W} = 0$. We denote growing solutions

$$(2.9) \quad \tilde{\Phi}^0 = (\tilde{\phi}_1^0, \dots, \tilde{\phi}_k^0), \quad \tilde{\Phi}^+ = (\tilde{\phi}_{k+1}^+, \dots, \tilde{\phi}_{n+r}^+),$$

$\tilde{\Phi} := (\tilde{\Phi}^0, \tilde{\Phi}^+)$ and decaying solutions

$$(2.10) \quad \tilde{\Psi}^0 = (\tilde{\psi}_1^0, \dots, \tilde{\psi}_k^+), \quad \tilde{\Psi}^+ = (\tilde{\psi}_{k+1}^+, \dots, \tilde{\psi}_{n+r}^+),$$

and $\tilde{\Psi} := (\tilde{\Psi}^0, \tilde{\Psi}^+)$, satisfying the relations

$$(2.11) \quad (\tilde{\Psi} \quad \tilde{\Phi})_{0,+}^* \tilde{\mathcal{S}}^{\tilde{\xi}} (\Psi \quad \Phi)_{0,+} \equiv I,$$

where

$$(2.12) \quad \tilde{\mathcal{S}}^{\tilde{\xi}} = \begin{pmatrix} -A^1 + iB^{1\tilde{\xi}} + iB^{\tilde{\xi}1} & \begin{pmatrix} 0 \\ I_r \end{pmatrix} \\ \begin{pmatrix} -(b_2^{11})^{-1}b_I^{11} & -I_r \end{pmatrix} & 0 \end{pmatrix}.$$

With these preparations, the construction of the Resolvent kernel goes exactly as in the construction performed in [ZH, MaZ3, Z3] on the whole line and [YZ, NZ] on the half line, yielding the following basic representation formulae; for a proof, see [MaZ3, NZ].

Proposition 2.1. *We have the following representation*

$$(2.13) \quad G_{\tilde{\xi}, \lambda}(x_1, y_1) = \begin{cases} (I_n, 0) \mathcal{F}^{y_1 \rightarrow x_1} \Pi_{y_1}^+ (\tilde{S}^{\tilde{\xi}})^{-1}(y_1) (I_n, 0)^{tr}, & \text{for } x_1 > y_1, \\ -(I_n, 0) \mathcal{F}^{y_1 \rightarrow x_1} \Pi_{y_1}^0 (\tilde{S}^{\tilde{\xi}})^{-1}(y_1) (I_n, 0)^{tr}, & \text{for } x_1 < y_1. \end{cases}$$

Proposition 2.2. *The resolvent kernel may alternatively be expressed as*

$$G_{\tilde{\xi}, \lambda}(x_1, y_1) = \begin{cases} (I_n, 0) \Phi^+(x_1; \lambda) M^+(\lambda) \tilde{\Psi}^{0*}(y_1; \lambda) (I_n, 0)^{tr} & x_1 > y_1, \\ -(I_n, 0) \Phi^0(x_1; \lambda) M^0(\lambda) \tilde{\Psi}^{+*}(y_1; \lambda) (I_n, 0)^{tr} & x_1 < y_1, \end{cases}$$

where

$$(2.14) \quad M(\lambda) := \text{diag}(M^+(\lambda), M^0(\lambda)) = \Phi^{-1}(z; \lambda) (\tilde{S}^{\tilde{\xi}})^{-1}(z) \tilde{\Psi}^{-1*}(z; \lambda).$$

2.1.3. Scattering decomposition. From Propositions 2.1 and 2.2, we obtain the following scattering decomposition, generalizing the Fourier transform representation in the constant-coefficient case, from which we will obtain pointwise bounds in the low-frequency regime.

Corollary 2.3. *On $\Lambda^{\tilde{\xi}} \cap \rho(L_{\tilde{\xi}})$,*

$$(2.15) \quad G_{\tilde{\xi}, \lambda}(x_1, y_1) = \sum_{j,k} d_{jk}^+ \phi_j^+(x_1; \lambda) \tilde{\psi}_k^+(y_1; \lambda)^* + \sum_k \phi_k^+(x_1; \lambda) \tilde{\phi}_k^+(y_1; \lambda)^*$$

for $0 \leq y_1 \leq x_1$, and

$$(2.16) \quad G_{\tilde{\xi}, \lambda}(x_1, y_1) = \sum_{j,k} d_{jk}^0 \phi_j^+(x_1; \lambda) \tilde{\psi}_k^+(y_1; \lambda)^* - \sum_k \psi_k^+(x_1; \lambda) \tilde{\psi}_k^+(y_1; \lambda)^*$$

for $0 \leq x_1 \leq y_1$, where

$$(2.17) \quad d_{jk}^{0,+}(\lambda) = (I, 0) (\Phi^+ \quad \Phi^0)^{-1} \Psi^+.$$

Proof. For $0 \leq x_1 \leq y_1$, we obtain the preliminary representation

$$G_{\tilde{\xi}, \lambda}(x_1, y_1) = \sum_{j,k} d_{jk}^0(\lambda) \phi_j^+(x_1; \lambda) \tilde{\psi}_k^+(y_1; \lambda)^* + \sum_{jk} e_{jk}^0 \psi_j^+(x_1; \lambda) \tilde{\psi}_k^+(y_1; \lambda)^*$$

from which, together with duality (2.11), representation (2.13), and the fact that $\Pi_0 = I - \Pi_+$, we have

$$(2.18) \quad \begin{aligned} \begin{pmatrix} d^0 \\ e^0 \end{pmatrix} &= -(\tilde{\Phi}^+ \quad \tilde{\Psi}^+)^* A \Pi_0 \Psi^+ \\ &= -(\Phi^+ \quad \Psi^+)^{-1} \left[I - (\Phi^+ \quad 0) (\Phi^+ \quad \Phi^0)^{-1} \right] \Psi^+ \\ &= \begin{pmatrix} 0 \\ -I_k \end{pmatrix} + \begin{pmatrix} I_{n-k} & 0 \\ 0 & 0 \end{pmatrix} (\Phi^+ \quad \Phi^0)^{-1} \Psi^+. \end{aligned}$$

Similarly, for $0 \leq y_1 \leq x_1$, we obtain the preliminary representation

$$G_{\tilde{\xi}, \lambda}(x_1, y_1) = \sum_{j,k} d_{jk}^+(\lambda) \phi_j^+(x_1; \lambda) \tilde{\psi}_k^+(y_1; \lambda)^* + \sum_{jk} e_{jk}^+ \phi_j^+(x_1; \lambda) \tilde{\phi}_k^+(y_1; \lambda)^*$$

from which, together with duality (2.11) and representation (2.13), we have

$$\begin{aligned}
 \begin{pmatrix} d^+ \\ e^+ \end{pmatrix} &= \tilde{\Phi}^{+*} A \Pi_+ (\Psi^+ \quad \Phi^+) \\
 &= (\Phi^+)^{-1} (\Phi^+ \quad 0) (\Phi^+ \quad \Phi^0)^{-1} (\Psi^+ \quad \Phi^+) \\
 &= (I \quad 0) (\Phi^+ \quad \Phi^0)^{-1} (\Psi^+ \quad \Phi^+) \\
 &= \begin{pmatrix} I_{n-k} & 0 \\ 0 & 0 \end{pmatrix} (\Phi^+ \quad \Phi^0)^{-1} \Psi^+ + \begin{pmatrix} 0 & 0 \\ I_k & 0 \end{pmatrix} \begin{pmatrix} 0 & I_k \\ 0 & 0 \end{pmatrix}.
 \end{aligned}
 \tag{2.19}$$

□

Remark 2.4. In the constant-coefficient case, with a choice of common bases $\Psi^{0,+} = \Phi^{+,0}$ at $0, +\infty$, the above representation reduces to the simple formula

$$G_{\tilde{\xi}, \lambda}(x_1, y_1) = \begin{cases} \sum_{j=k+1}^N \phi_j^+(x_1; \lambda) \tilde{\phi}_j^{+*}(y_1; \lambda) & x_1 > y_1, \\ -\sum_{j=1}^k \psi_j^+(x_1; \lambda) \tilde{\psi}_j^{+*}(y_1; \lambda) & x_1 < y_1. \end{cases}
 \tag{2.20}$$

2.2. Pointwise low-frequency bounds. We obtain pointwise low-frequency bounds on the resolvent kernel $G_{\tilde{\xi}, \lambda}(x_1, y_1)$ by appealing to the detailed analysis of [Z2, Z3, GMWZ1] in the viscous shock case. Restrict attention to the surface

$$\Gamma^{\tilde{\xi}} := \{\lambda : \Re \lambda = -\theta_1(|\tilde{\xi}|^2 + |\Im m \lambda|^2)\},
 \tag{2.21}$$

for $\theta_1 > 0$ sufficiently small.

Proposition 2.5 ([Z3]). *Under the hypotheses of Theorem 1.7, for $\lambda \in \Gamma^{\tilde{\xi}}$ and $\rho := |(\tilde{\xi}, \lambda)|$, $\theta_1 > 0$, and $\theta > 0$ sufficiently small, there hold:*

$$|G_{\tilde{\xi}, \lambda}(x_1, y_1)| \leq C \gamma_2 e^{-\theta \rho^2 |x_1 - y_1|}.
 \tag{2.22}$$

and

$$|\partial_{y_1}^\beta G_{\tilde{\xi}, \lambda}(x_1, y_1)| \leq C \gamma_2 (\rho^\beta + \beta e^{-\theta y_1}) e^{-\theta \rho^2 |x_1 - y_1|}
 \tag{2.23}$$

where

$$\gamma_2 := 1 + \sum_j \left[\rho^{-1} |\Im m \lambda - \eta_j^+(\tilde{\xi})| + \rho \right]^{1/s_j - 1},
 \tag{2.24}$$

and $s_j, \eta_j^+(\tilde{\xi})$ are as defined in (H4).

Proof. This follows by a simplified version of the analysis of [Z3], Section 5 in the viscous shock case, replacing Φ^-, Ψ^- with Φ^0, Ψ^0 , omitting the refined derivative bounds of Lemmas 5.23 and 5.27 describing special properties of the Lax and overcompressive shock case (not relevant here), and setting $\ell = 0$, or $\tilde{\gamma} \equiv 1$ in definition (5.128). Here, ℓ is the multiplicity to which the Evans function vanishes at the origin, $(\tilde{\xi}, \lambda) = (0, 0)$, evidently zero under assumption (D). The key modes Φ^+, Ψ^+ at plus spatial infinity are the same for the boundary-layer as for the shock case.

This leads to the pointwise bounds (5.37)–(5.38) given in Proposition 5.10 of [Z3] in case $\alpha = 1$, $\gamma_1 \equiv 1$ corresponding to the uniformly stable undercompressive shock case, but

without the first $O(\rho^{-1})$, or “pole”, terms appearing on the righthand side, which derive from cases $\tilde{\gamma} \sim \rho^{-1}$ not arising here. But, these are exactly the claimed bounds (2.22)–(2.24).

We omit the (substantial) details of this computation, referring the reader to [Z3]. However, the basic idea is, starting with the scattering decomposition of Corollary 2.1.3, to note, first, that the normal modes Φ^j , Ψ^j , $\tilde{\Phi}^j$, $\tilde{\Psi}^j$ can be approximated up to an exponentially trivial coordinate change by solutions of the constant-coefficient limiting system at $x \rightarrow +\infty$ (the conjugation lemma of [MZ]) and, second, that the coefficients M_{jk} , d_{jk} may be well-estimated through formulae (2.14) and (2.17) using Kramer’s rule and the assumed lower bound on the Evans function $|D|$ appearing in the denominator. This is relatively straightforward away from the branch points $\Im \lambda = \eta_j(\xi)$ or “glancing set” of hyperbolic theory; the treatment near these points involves some delicate matrix perturbation theory applied to the limiting constant-coefficient system at $x \rightarrow +\infty$ followed by careful bookkeeping in the application of Kramer’s rule. \square

3. LINEARIZED ESTIMATES

We next establish estimates on the linearized inhomogeneous problem

$$(3.1) \quad U_t - LU = f$$

with initial data $U(0) = U_0$ and Dirichlet boundary conditions as usual in \tilde{W} -coordinates:

$$(3.2) \quad W(0, \tilde{x}, t) := (w^I, w^{II})^T(0, \tilde{x}, t) = h$$

for the inflow case, and

$$(3.3) \quad w^{II}(0, \tilde{x}, t) = h$$

for the outflow case, with $x = (x_1, \tilde{x}) \in \mathbb{R}^d$.

3.1. Resolvent bounds. Our first step is to estimate solutions of the resolvent equation with homogeneous boundary data $\hat{h} \equiv 0$.

Proposition 3.1 (High-frequency bounds). *Given (A1)–(A2), (H0)–(H2), and homogeneous boundary conditions (B), for some R, C sufficiently large and $\theta > 0$ sufficiently small,*

$$(3.4) \quad |(L_{\tilde{\xi}} - \lambda)^{-1} \hat{f}|_{\hat{H}^1(x_1)} \leq C |\hat{f}|_{\hat{H}^1(x_1)},$$

and

$$(3.5) \quad |(L_{\tilde{\xi}} - \lambda)^{-1} \hat{f}|_{L^2(x_1)} \leq \frac{C}{|\lambda|^{1/2}} |\hat{f}|_{\hat{H}^1(x_1)},$$

for all $|(\tilde{\xi}, \lambda)| \geq R$ and $\Re \lambda \geq -\theta$, where \hat{f} is the Fourier transform of f in variable \tilde{x} and $|\hat{f}|_{\hat{H}^1(x_1)} := |(1 + |\partial_{x_1}| + |\tilde{\xi}|) \hat{f}|_{L^2(x_1)}$.

Proof. First observe that a Laplace-Fourier transformed version with respect to variables (λ, \tilde{x}) of the nonlinear energy estimate in Section 4.1 with $s = 1$, carried out on the linearized equations written in W -coordinates, yields

$$(3.6) \quad (\Re \lambda + \theta_1) |(1 + |\tilde{\xi}| + |\partial_{x_1}|) W|^2 \leq C \left(|W|^2 + (1 + |\tilde{\xi}|^2) |W| |\hat{f}| + |\partial_{x_1} W| |\partial_{x_1} \hat{f}| \right)$$

for some C big and $\theta_1 > 0$ sufficiently small, where $|\cdot|$ denotes $|\cdot|_{L^2(x_1)}$. Applying Young's inequality, we obtain

$$(3.7) \quad (\Re \lambda + \theta_1) |(1 + |\tilde{\xi}| + |\partial_{x_1}|)W|^2 \leq C|W|^2 + C|(1 + |\tilde{\xi}| + |\partial_{x_1}|)\hat{f}|^2.$$

On the other hand, taking the imaginary part of the L^2 inner product of U against $\lambda U = f + LU$, we have also the standard estimate

$$(3.8) \quad |\Im \lambda| |U|_{L^2}^2 \leq C|U|_{H^1}^2 + C|f|_{L^2}^2,$$

and thus, taking the Fourier transform in \tilde{x} , we obtain

$$(3.9) \quad |\Im \lambda| |W|^2 \leq C|\hat{f}|^2 + C|(1 + |\tilde{\xi}| + |\partial_{x_1}|)W|^2.$$

Therefore, taking $\theta = \theta_1/2$, we obtain from (3.7) and (3.9)

$$(3.10) \quad |(1 + |\lambda|^{1/2} + |\tilde{\xi}| + |\partial_{x_1}|)W|^2 \leq C|W|^2 + C|(1 + |\tilde{\xi}| + |\partial_{x_1}|)\hat{f}|^2,$$

for any $\Re \lambda \geq -\theta$. Now take R sufficiently large such that $|W|^2$ on the right hand side of the above can be absorbed into the left hand side, and thus, for all $|(\tilde{\xi}, \lambda)| \geq R$ and $\Re \lambda \geq -\theta$,

$$(3.11) \quad |(1 + |\lambda|^{1/2} + |\tilde{\xi}| + |\partial_{x_1}|)W|^2 \leq C|(1 + |\tilde{\xi}| + |\partial_{x_1}|)\hat{f}|^2,$$

for some large $C > 0$, which gives the result. \square

We next have the following:

Proposition 3.2 (Mid-frequency bounds). *Given (A1)-(A2), (H0)-(H2), and strong spectral stability (D),*

$$(3.12) \quad |(L_{\tilde{\xi}} - \lambda)^{-1}|_{\dot{H}^1(x_1)} \leq C, \quad \text{for } R^{-1} \leq |(\tilde{\xi}, \lambda)| \leq R \text{ and } \Re \lambda \geq -\theta,$$

for any R and $C = C(R)$ sufficiently large and $\theta = \theta(R) > 0$ sufficiently small, where $|\hat{f}|_{\dot{H}^1(x_1)}$ is defined as in Proposition 3.1.

Proof. Immediate, by compactness of the set of frequencies under consideration together with the fact that the resolvent $(\lambda - L_{\tilde{\xi}})^{-1}$ is analytic with respect to H^1 in $(\tilde{\xi}, \lambda)$; see Proposition 4.8, [Z4]. \square

We next obtain the following resolvent bound for low-frequency regions as a direct consequence of pointwise bounds on the resolvent kernel, obtained in Proposition 2.5.

Proposition 3.3 (Low-frequency bounds). *Under the hypotheses of Theorem 1.7, for $\lambda \in \Gamma^{\tilde{\xi}}$ and $\rho := |(\tilde{\xi}, \lambda)|$, θ_1 sufficiently small, there holds the resolvent bound*

$$(3.13) \quad |(L_{\tilde{\xi}} - \lambda)^{-1} \partial_{x_1}^{\beta} \hat{f}|_{L^p(x_1)} \leq C \gamma_2 \rho^{-2/p} \left[\rho^{\beta} |\hat{f}|_{L^1(x_1)} + \beta |\hat{f}|_{L^{\infty}(x_1)} \right]$$

for all $2 \leq p \leq \infty$, $\beta = 0, 1$, where γ_2 is as defined in (2.24).

Proof. Using the convolution inequality $|g * h|_{L^p} \leq |g|_{L^p} |h|_{L^1}$ and noticing that

$$|\partial_{y_1}^{\beta} G_{\tilde{\xi}, \lambda}(x_1, y_1)| \leq C \gamma_2 (\rho^{\beta} + \beta e^{-\theta y_1}) e^{-\theta \rho^2 |x_1 - y_1|},$$

we obtain

$$\begin{aligned}
 |(L_{\tilde{\xi}} - \lambda)^{-1} \partial_{x_1}^\beta \hat{f}|_{L^p(x_1)} &= \left| \int \partial_{y_1}^\beta G_{\tilde{\xi}, \lambda}(x_1, y_1) \hat{f}(y_1, \tilde{\xi}) dy_1 \right|_{L^p(x_1)} \\
 (3.14) \quad &\leq \left| \int C\gamma_2(\rho^\beta + \beta e^{-\theta y_1}) e^{-\theta \rho^2 |x_1 - y_1|} |\hat{f}(y_1, \tilde{\xi})| dy_1 \right|_{L^p} \\
 &\leq C\gamma_2 \rho^{-2/p} \left[\rho^\beta |\hat{f}|_{L^1(x_1)} + \beta |\hat{f}|_{L^\infty(x_1)} \right]
 \end{aligned}$$

as claimed. \square

Remark 3.4. The above L^p bounds may alternatively be obtained directly by the argument of Section 12, [GMWZ1], using quite different Kreiss symmetrizer techniques, again omitting pole terms arising from vanishing of the Evans function at the origin, and also the auxiliary problem construction of Section 12.6 used to obtain sharpened bounds in the Lax or overcompressive shock case (not relevant here).

3.2. Estimates on homogeneous solution operators. Define low- and high-frequency parts of the linearized solution operator $\mathcal{S}(t)$ of the linearized problem with homogeneous boundary and forcing data, $f, h \equiv 0$, as

$$(3.15) \quad \mathcal{S}_1(t) := \frac{1}{(2\pi i)^d} \int_{|\tilde{\xi}| \leq r} \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\lambda t + i\tilde{\xi} \cdot \tilde{x}} (L_{\tilde{\xi}} - \lambda)^{-1} d\lambda d\tilde{\xi}$$

and

$$(3.16) \quad \mathcal{S}_2(t) := e^{Lt} - \mathcal{S}_1(t).$$

Then we obtain the following:

Proposition 3.5 (Low-frequency estimate). *Under the hypotheses of Theorem 1.7, for $\beta = (\beta_1, \beta')$ with $\beta_1 = 0, 1$,*

$$\begin{aligned}
 |\mathcal{S}_1(t) \partial_x^\beta f|_{L_x^2} &\leq C(1+t)^{-(d-1)/4 - |\beta|/2} |f|_{L_x^1} + C\beta_1(1+t)^{-(d-1)/4} |f|_{L_{\tilde{x}, x_1}^{1, \infty}}, \\
 (3.17) \quad |\mathcal{S}_1(t) \partial_x^\beta f|_{L_{\tilde{x}, x_1}^{2, \infty}} &\leq C(1+t)^{-(d+1)/4 - |\beta|/2} |f|_{L_x^1} + C\beta_1(1+t)^{-(d+1)/4} |f|_{L_{\tilde{x}, x_1}^{1, \infty}}, \\
 |\mathcal{S}_1(t) \partial_x^\beta f|_{L_{\tilde{x}, x_1}^\infty} &\leq C(1+t)^{-d/2 - |\beta|/2} |f|_{L_x^1} + C\beta_1(1+t)^{-d/2} |f|_{L_{\tilde{x}, x_1}^{1, \infty}},
 \end{aligned}$$

where $|\cdot|_{L_{\tilde{x}, x_1}^{p, q}}$ denotes the norm in $L^p(\tilde{x}; L^q(x_1))$.

Proof. The proof will follow closely the treatment of the shock case in [Z3]. Let $\hat{u}(x_1, \tilde{\xi}, \lambda)$ denote the solution of $(L_{\tilde{\xi}} - \lambda)\hat{u} = \hat{f}$, where $\hat{f}(x_1, \tilde{\xi})$ denotes Fourier transform of f , and

$$u(x, t) := \mathcal{S}_1(t)f = \frac{1}{(2\pi i)^d} \int_{|\tilde{\xi}| \leq r} \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\lambda t + i\tilde{\xi} \cdot \tilde{x}} (L_{\tilde{\xi}} - \lambda)^{-1} \hat{f}(x_1, \tilde{\xi}) d\lambda d\tilde{\xi}.$$

Recalling the resolvent estimates in Proposition 3.3, we have

$$|\hat{u}(x_1, \tilde{\xi}, \lambda)|_{L^p(x_1)} \leq C\gamma_2 \rho^{-2/p} |\hat{f}|_{L^1(x_1)} \leq C\gamma_2 \rho^{-2/p} |f|_{L^1(x)}$$

where γ_2 is as defined in (2.24).

Therefore, using Parseval's identity, Fubini's theorem, and the triangle inequality, we may estimate

$$\begin{aligned}
|u|_{L^2(x_1, \tilde{x})}^2(t) &= \frac{1}{(2\pi)^{2d}} \int_{x_1} \int_{\tilde{\xi}} \left| \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\lambda t} \hat{u}(x_1, \tilde{\xi}, \lambda) d\lambda \right|^2 d\tilde{\xi} dx_1 \\
&= \frac{1}{(2\pi)^{2d}} \int_{\tilde{\xi}} \left| \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\lambda t} \hat{u}(x_1, \tilde{\xi}, \lambda) d\lambda \right|_{L^2(x_1)}^2 d\tilde{\xi} \\
&\leq \frac{1}{(2\pi)^{2d}} \int_{\tilde{\xi}} \left| \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\Re \lambda t} |\hat{u}(x_1, \tilde{\xi}, \lambda)|_{L^2(x_1)} d\lambda \right|^2 d\tilde{\xi} \\
&\leq C |f|_{L^1(x)}^2 \int_{\tilde{\xi}} \left| \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\Re \lambda t} \gamma_2 \rho^{-1} d\lambda \right|^2 d\tilde{\xi}.
\end{aligned}$$

Specifically, parametrizing $\Gamma^{\tilde{\xi}}$ by

$$\lambda(\tilde{\xi}, k) = ik - \theta_1(k^2 + |\tilde{\xi}|^2), \quad k \in \mathbb{R},$$

and observing that by (2.24),

$$\begin{aligned}
(3.18) \quad \gamma_2 \rho^{-1} &\leq (|k| + |\tilde{\xi}|)^{-1} \left[1 + \sum_j \left(\frac{|k - \tau_j(\tilde{\xi})|}{\rho} \right)^{1/s_j - 1} \right] \\
&\leq (|k| + |\tilde{\xi}|)^{-1} \left[1 + \sum_j \left(\frac{|k - \tau_j(\tilde{\xi})|}{\rho} \right)^{\epsilon - 1} \right],
\end{aligned}$$

where $\epsilon := \frac{1}{\max_j s_j}$ ($0 < \epsilon < 1$ chosen arbitrarily if there are no singularities), we estimate

$$\begin{aligned}
\int_{\tilde{\xi}} \left| \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\Re \lambda t} \gamma_2 \rho^{-1} d\lambda \right|^2 d\tilde{\xi} &\leq \int_{\tilde{\xi}} \left| \int_{\mathbb{R}} e^{-\theta_1(k^2 + |\tilde{\xi}|^2)t} \gamma_2 \rho^{-1} dk \right|^2 d\tilde{\xi} \\
&\leq \int_{\tilde{\xi}} e^{-2\theta_1|\tilde{\xi}|^2 t} |\tilde{\xi}|^{-2\epsilon} \left| \int_{\mathbb{R}} e^{-\theta_1 k^2 t} |k|^{\epsilon-1} dk \right|^2 d\tilde{\xi} \\
&\quad + \sum_j \int_{\tilde{\xi}} e^{-2\theta_1|\tilde{\xi}|^2 t} |\tilde{\xi}|^{-2\epsilon} \left| \int_{\mathbb{R}} e^{-\theta_1 k^2 t} |k - \tau_j(\tilde{\xi})|^{\epsilon-1} dk \right|^2 d\tilde{\xi} \\
&\leq \int_{\tilde{\xi}} e^{-2\theta_1|\tilde{\xi}|^2 t} |\tilde{\xi}|^{-2\epsilon} \left| \int_{\mathbb{R}} e^{-\theta_1 k^2 t} |k|^{\epsilon-1} dk \right|^2 d\tilde{\xi} \\
&\leq C t^{-(d-1)/2}.
\end{aligned}$$

Likewise, we have

$$\begin{aligned}
|u|_{L_{\tilde{x}, x_1}^{2, \infty}}^2(t) &= \frac{1}{(2\pi)^{2d}} \int_{\tilde{\xi}} \left| \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\lambda t} \hat{u}(x_1, \tilde{\xi}, \lambda) d\lambda \right|_{L^\infty(x_1)}^2 d\tilde{\xi} \\
&\leq \frac{1}{(2\pi)^{2d}} \int_{\tilde{\xi}} \left| \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\Re \lambda t} |\hat{u}(x_1, \tilde{\xi}, \lambda)|_{L^\infty(x_1)} d\lambda \right|^2 d\tilde{\xi} \\
&\leq C |f|_{L^1(x)}^2 \int_{\tilde{\xi}} \left| \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\Re \lambda t} \gamma_2 d\lambda \right|^2 d\tilde{\xi}
\end{aligned}$$

where

$$\begin{aligned}
\int_{\tilde{\xi}} \left| \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\Re \lambda t} \gamma_2 d\lambda \right|^2 d\tilde{\xi} &\leq \int_{\tilde{\xi}} e^{-2\theta_1 |\tilde{\xi}|^2 t} \left| \int_{\mathbb{R}} e^{-\theta_1 k^2 t} dk \right|^2 d\tilde{\xi} \\
&\quad + \sum_j \int_{\tilde{\xi}} e^{-2\theta_1 |\tilde{\xi}|^2 t} |\tilde{\xi}|^{2-2\epsilon} \left| \int_{\mathbb{R}} e^{-\theta_1 k^2 t} |k - \tau_j(\tilde{\xi})|^{\epsilon-1} dk \right|^2 d\tilde{\xi} \\
&\leq C t^{-(d+1)/2} + C \int_{\tilde{\xi}} e^{-2\theta_1 |\tilde{\xi}|^2 t} |\tilde{\xi}|^{2-2\epsilon} \left| \int_{\mathbb{R}} e^{-\theta_1 k^2 t} |k|^{\epsilon-1} dk \right|^2 d\tilde{\xi} \\
&\leq C t^{-(d+1)/2}.
\end{aligned}$$

Similarly, we estimate

$$\begin{aligned}
|u|_{L_{\tilde{x}, x_1}^\infty}(t) &\leq \frac{1}{(2\pi)^d} \int_{\tilde{\xi}} \left| \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\lambda t} \hat{u}(x_1, \tilde{\xi}, \lambda) d\lambda \right|_{L^\infty(x_1)} d\tilde{\xi} \\
&\leq \frac{1}{(2\pi)^d} \int_{\tilde{\xi}} \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\Re \lambda t} |\hat{u}(x_1, \tilde{\xi}, \lambda)|_{L^\infty(x_1)} d\lambda d\tilde{\xi} \\
&\leq C |f|_{L^1(x)} \int_{\tilde{\xi}} \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\Re \lambda t} \gamma_2 d\lambda d\tilde{\xi}
\end{aligned}$$

where as above we have

$$\begin{aligned}
\int_{\tilde{\xi}} \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\Re \lambda t} \gamma_2 d\lambda d\tilde{\xi} &\leq \int_{\tilde{\xi}} e^{-\theta_1 |\tilde{\xi}|^2 t} \int_{\mathbb{R}} e^{-\theta_1 k^2 t} dk d\tilde{\xi} \\
&\quad + \sum_j \int_{\tilde{\xi}} e^{-\theta_1 |\tilde{\xi}|^2 t} |\tilde{\xi}|^{1-\epsilon} \int_{\mathbb{R}} e^{-\theta_1 k^2 t} |k - \tau_j(\tilde{\xi})|^{\epsilon-1} dk d\tilde{\xi} \\
&\leq C t^{-d/2} + C \int_{\tilde{\xi}} e^{-\theta_1 |\tilde{\xi}|^2 t} |\tilde{\xi}|^{1-\epsilon} \int_{\mathbb{R}} e^{-\theta_1 k^2 t} |k|^{\epsilon-1} dk d\tilde{\xi} \\
&\leq C t^{-d/2}.
\end{aligned}$$

The x_1 -derivative bounds follow similarly by using the resolvent bounds in Proposition 3.3 with $\beta_1 = 1$. The \tilde{x} -derivative bounds are straightforward by the fact that $\widehat{\partial_{\tilde{x}}^{\tilde{\beta}} f} = (i\tilde{\xi})^{\tilde{\beta}} \hat{f}$.

Finally, each of the above integrals is bounded by $C|f|_{L^1(x)}$ as the product of $|f|_{L^1(x)}$ times the integral quantities $\gamma_2 \rho^{-1}$, γ_2 over a bounded domain, hence we may replace t by $(1+t)$ in the above estimates. \square

Next, we obtain estimates on the high-frequency part $\mathcal{S}_2(t)$ of the linearized solution operator. Recall that $\mathcal{S}_2(t) = \mathcal{S}(t) - \mathcal{S}_1(t)$, where

$$\mathcal{S}(t) = \frac{1}{(2\pi i)^d} \int_{\mathbb{R}^{d-1}} e^{i\tilde{\xi} \cdot \tilde{x}} e^{L_{\tilde{\xi}} t} d\tilde{\xi}$$

and

$$\mathcal{S}_1(t) = \frac{1}{(2\pi i)^d} \int_{|\tilde{\xi}| \leq r} \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\lambda t + i\tilde{\xi} \cdot \tilde{x}} (L_{\tilde{\xi}} - \lambda)^{-1} d\lambda d\tilde{\xi}.$$

Then according to [Z4, Corollary 4.11], we can write

$$(3.19) \quad \begin{aligned} \mathcal{S}_2(t)f &= \frac{1}{(2\pi i)^d} \text{P.V.} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} \int_{\mathbb{R}^{d-1}} \chi_{|\tilde{\xi}|^2+|\Im m \lambda|^2 \geq \theta_1+\theta_2} \\ &\quad \times e^{i\tilde{\xi} \cdot \tilde{x} + \lambda t} (\lambda - L_{\tilde{\xi}})^{-1} \hat{f}(x_1, \tilde{\xi}) d\tilde{\xi} d\lambda. \end{aligned}$$

Proposition 3.6 (High-frequency estimate). *Given (A1)-(A2), (H0)-(H2), (D), and homogeneous boundary conditions (B), for $0 \leq |\alpha| \leq s-3$, s as in (H0),*

$$(3.20) \quad \begin{aligned} |\mathcal{S}_2(t)f|_{L_x^2} &\leq C e^{-\theta_1 t} |f|_{H_x^3}, \\ |\partial_x^\alpha \mathcal{S}_2(t)f|_{L_x^2} &\leq C e^{-\theta_1 t} |f|_{H_x^{|\alpha|+3}}. \end{aligned}$$

Proof. The proof starts with the following resolvent identity, using analyticity on the resolvent set $\rho(L_{\tilde{\xi}})$ of the resolvent $(\lambda - L_{\tilde{\xi}})^{-1}$, for all $f \in \mathcal{D}(L_{\tilde{\xi}})$,

$$(3.21) \quad (\lambda - L_{\tilde{\xi}})^{-1} f = \lambda^{-1} (\lambda - L_{\tilde{\xi}})^{-1} L_{\tilde{\xi}} f + \lambda^{-1} f.$$

Using this identity and (3.19), we estimate

$$(3.22) \quad \begin{aligned} \mathcal{S}_2(t)f &= \frac{1}{(2\pi i)^d} \text{P.V.} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} \int_{\mathbb{R}^{d-1}} \chi_{|\tilde{\xi}|^2+|\Im m \lambda|^2 \geq \theta_1+\theta_2} \\ &\quad \times e^{i\tilde{\xi} \cdot \tilde{x} + \lambda t} \lambda^{-1} (\lambda - L_{\tilde{\xi}})^{-1} L_{\tilde{\xi}} \hat{f}(x_1, \tilde{\xi}) d\tilde{\xi} d\lambda \\ &\quad + \frac{1}{(2\pi i)^d} \text{P.V.} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} \int_{\mathbb{R}^{d-1}} \chi_{|\tilde{\xi}|^2+|\Im m \lambda|^2 \geq \theta_1+\theta_2} \\ &\quad \times e^{i\tilde{\xi} \cdot \tilde{x} + \lambda t} \lambda^{-1} \hat{f}(x_1, \tilde{\xi}) d\tilde{\xi} d\lambda \\ &=: S_1 + S_2, \end{aligned}$$

where, by Plancherel's identity and Propositions 3.6 and 3.2, we have

$$\begin{aligned} |S_1|_{L^2(\tilde{x}, x_1)} &\leq C \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} |\lambda|^{-1} |e^{\lambda t}| |(\lambda - L_{\tilde{\xi}})^{-1} L_{\tilde{\xi}} \hat{f}|_{L^2(\tilde{\xi}, x_1)} |d\lambda| \\ &\leq C e^{-\theta_1 t} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} |\lambda|^{-3/2} \left| (1 + |\tilde{\xi}|) |L_{\tilde{\xi}} \hat{f}|_{H^1(x_1)} \right|_{L^2(\tilde{\xi})} |d\lambda| \\ &\leq C e^{-\theta_1 t} |f|_{H_x^3} \end{aligned}$$

and

$$(3.23) \quad \begin{aligned} |S_2|_{L_x^2} &\leq \frac{1}{(2\pi)^d} \left| \text{P.V.} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} \lambda^{-1} e^{\lambda t} d\lambda \int_{\mathbb{R}^{d-1}} e^{i\tilde{x} \cdot \tilde{\xi}} \hat{f}(x_1, \tilde{\xi}) d\tilde{\xi} \right|_{L^2} \\ &\quad + \frac{1}{(2\pi)^d} \left| \text{P.V.} \int_{-\theta_1-ir}^{-\theta_1+ir} \lambda^{-1} e^{\lambda t} d\lambda \int_{\mathbb{R}^{d-1}} e^{i\tilde{x} \cdot \tilde{\xi}} \hat{f}(x_1, \tilde{\xi}) d\tilde{\xi} \right|_{L^2} \\ &\leq C e^{-\theta_1 t} |f|_{L_x^2}, \end{aligned}$$

by direct computations, noting that the integral in λ in the first term is identically zero. This completes the proof of the first inequality stated in the proposition. Derivative bounds follow similarly. \square

Remark 3.7. Here, we have used the $\lambda^{1/2}$ improvement in (3.5) over (3.4) together with modifications introduced in [KZ] to greatly simplify the original high-frequency argument given in [Z3] for the shock case.

3.3. Boundary estimates. For the purpose of studying the nonzero boundary perturbation, we need the following proposition. For $h := h(\tilde{x}, t)$, define

$$(3.24) \quad \mathcal{D}_h(t) := (|h_t| + |h_{\tilde{x}}| + |h_{\tilde{x}\tilde{x}}|)(t),$$

and

$$(3.25) \quad \Gamma h(t) := \int_0^t \int_{\mathbb{R}^{d-1}} \left(\sum_k G_{y_k} B^{k1} + GA^1 \right)(x, t-s; 0, \tilde{y}) h(\tilde{y}, s) d\tilde{y} ds,$$

where $G(x, t; y)$ is the Green function of $\partial_t - L$. This boundary term will appear when we write down the Duhamel formulas for the linearized and nonlinear equations (see (3.37) and (4.55)). Noting that for the outflow case, the fact that $G(x, t; 0, \tilde{y}) \equiv 0$ simplifies Γh to

$$(3.26) \quad \Gamma h(t) = \int_0^t \int_{\mathbb{R}^{d-1}} G_{y_1}(x, t-s; 0, \tilde{y}) B^{11} h d\tilde{y} ds.$$

Therefore when dealing with the outflow case, instead of putting assumptions on h itself as in the inflow case, we make assumptions on $B^{11}h$, matching with the hypotheses on W -coordinates.

Proposition 3.8. *Assume that $h = h(\tilde{x}, t)$ satisfies*

$$(3.27) \quad \begin{aligned} |h(t)|_{L_{\tilde{x}}^2} &\leq E_0(1+t)^{-(d+1)/4}, \\ |h(t)|_{L_{\tilde{x}}^\infty} &\leq E_0(1+t)^{-d/2} \\ |\mathcal{D}_h(t)|_{L_{\tilde{x}}^1 \cap H_{\tilde{x}}^{|\gamma|+3}} &\leq E_0(1+t)^{-d/2-\epsilon}, \end{aligned}$$

for some positive constant E_0 ; here $|\gamma| = [(d-1)/2] + 2$, and $\epsilon > 0$ is arbitrary small for $d = 2$ and $\epsilon = 0$ for $d \geq 3$. For the outflow case, we replace these assumptions on h by those on $B^{11}h$. Then we obtain

$$(3.28) \quad \begin{aligned} |\Gamma h(t)|_{L^2} &\leq CE_0(1+t)^{-(d-1)/4}, \\ |\Gamma h(t)|_{L_{\tilde{x}, x_1}^{2, \infty}} &\leq CE_0(1+t)^{-(d+1)/4}, \\ |\Gamma h(t)|_{L^\infty} &\leq CE_0(1+t)^{-d/2}, \end{aligned}$$

and derivative bounds

$$(3.29) \quad \begin{aligned} |\partial_x \Gamma h(t)|_{L_{\tilde{x}, x_1}^{2, \infty}} &\leq CE_0(1+t)^{-(d+1)/4}, \\ |\partial_{\tilde{x}}^2 \Gamma h(t)|_{L_{\tilde{x}, x_1}^{2, \infty}} &\leq CE_0(1+t)^{-(d+1)/4}, \end{aligned}$$

for all $t \geq 0$.

Proof. We first recall that $G(x, t-s; y)$ is a solution of $(\partial_s - L_y)^* G^* = 0$, that is,

$$(3.30) \quad -G_s - \sum_j (GA^j)_{y_j} + \sum_j GA_{y_j}^j = \sum_{jk} (G_{y_k} B^{kj})_{y_j}.$$

Integrating this on $\mathbb{R}_+^d \times [0, t]$ against

$$(3.31) \quad g(y_1, \tilde{y}, s) := e^{-y_1} h(\tilde{y}, s),$$

and integrating by parts twice, we obtain

$$\begin{aligned} \Gamma h &= - \int_0^t \int_{\mathbb{R}_+^d} \left(\sum_{jk} G_{y_k} B^{kj} + \sum_j G A^j \right) g_{y_j} dy ds \\ &\quad - \int_0^t \int_{\mathbb{R}_+^d} \left(-G_s + \sum_j G A_{y_j}^j \right) g(y, s) dy ds \end{aligned}$$

where, recalling that

$$\mathcal{S}(t)f(x) = \int_{\mathbb{R}_+^d} G(x, t; y) f(y) dy,$$

we get

$$\begin{aligned} &- \int_0^t \int_{\mathbb{R}_+^d} \sum_{jk} \left(G_{y_k} B^{kj} + \sum_j G A^j \right) g_{y_j} dy ds \\ &= - \int_0^t \mathcal{S}(t-s) \left(- \sum_{jk} (B^{kj} g_{x_j})_{x_k} + \sum_j A^j g_{x_j} \right) ds \end{aligned}$$

and

$$\begin{aligned} &- \int_0^t \int_{\mathbb{R}_+^d} \left(-G_s + \sum_j G A_{y_j}^j \right) g(y, s) dy ds \\ &= - \int_0^t \mathcal{S}(t-s) \left(g_s + \sum_j A_{x_j}^j g \right) ds + g(x, t) - \mathcal{S}(t)g(x, 0). \end{aligned}$$

Therefore combining all these estimates yields

$$(3.32) \quad \Gamma h = g(x, t) - \mathcal{S}(t)g_0 - \int_0^t \mathcal{S}(t-s)(g_s - L_x g(x, s)) ds$$

with $g_0(x) := g(x, 0)$ and $L_x g = - \sum_j (A^j g)_{x_j} + \sum_{jk} (B^{jk} g_{x_k})_{x_j}$.

Now we are ready to employ estimates obtained in the previous section on the solution operator $\mathcal{S}(t) = \mathcal{S}_1(t) + \mathcal{S}_2(t)$. Noting that

$$|g|_{L_x^p} \leq C |h|_{L_x^p},$$

we estimate

$$\begin{aligned}
|\Gamma h|_{L^2} &\leq |g|_{L^2} + |\mathcal{S}_1(t)g_0|_{L^2} + |\mathcal{S}_2(t)g_0|_{L^2} \\
&\quad + \int_0^t |\mathcal{S}_1(t-s)(g_s - Lg)|_{L^2} + |\mathcal{S}_2(t-s)(g_s - Lg)|_{L^2} ds \\
&\leq |h(t)|_{L_{\tilde{x}}^2} + C(1+t)^{-\frac{d-1}{4}} |g_0|_{L^1} + Ce^{-\eta t} |g_0|_{H^3} \\
&\quad + \int_0^t (1+t-s)^{-(d-1)/4} (|g_s| + |Lg|)_{L^1} + e^{-\theta(t-s)} (|g_s| + |Lg|)_{H^3} ds \\
&\leq |h(t)|_{L_{\tilde{x}}^2} + C(1+t)^{-\frac{d-1}{4}} |h_0|_{L_{\tilde{x}}^1 \cap H_{\tilde{x}}^3} \\
&\quad + \int_0^t (1+t-s)^{-(d-1)/4} |\mathcal{D}_h(s)|_{L_{\tilde{x}}^1} + e^{-\theta(t-s)} |\mathcal{D}_h(s)|_{H_{\tilde{x}}^3} ds \\
&\leq CE_0(1+t)^{-\frac{d-1}{4}}
\end{aligned}$$

and similarly we also obtain

$$\begin{aligned}
|\Gamma h|_{L_{\tilde{x},x_1}^{2,\infty}} &\leq |h(t)|_{L_{\tilde{x}}^2} + C(1+t)^{-\frac{d+1}{4}} |h_0|_{L_{\tilde{x}}^1 \cap H_{\tilde{x}}^4} \\
(3.33) \quad &\quad + C \int_0^t (1+t-s)^{-(d+1)/4} |\mathcal{D}_h(s)|_{L_{\tilde{x}}^1} + e^{-\theta(t-s)} |\mathcal{D}_h(s)|_{H_{\tilde{x}}^4} ds \\
&\leq CE_0(1+t)^{-\frac{d+1}{4}}
\end{aligned}$$

and

$$\begin{aligned}
|\Gamma h|_{L^\infty} &\leq |h(t)|_{L_{\tilde{x}}^\infty} + C(1+t)^{-\frac{d}{2}} |h_0|_{L_{\tilde{x}}^1 \cap H_{\tilde{x}}^{|\gamma|+3}} \\
(3.34) \quad &\quad + C \int_0^t (1+t-s)^{-d/2} |\mathcal{D}_h(s)|_{L_{\tilde{x}}^1} + e^{-\theta(t-s)} |\mathcal{D}_h(s)|_{H_{\tilde{x}}^{|\gamma|+3}} ds \\
&\leq CE_0(1+t)^{-\frac{d}{2}}.
\end{aligned}$$

Similar bounds hold for derivatives.

This completes the proof of the proposition. \square

3.4. Duhamel formula. The following integral representation formula expresses the solution of the inhomogeneous equation (3.1) in terms of the homogeneous solution operator \mathcal{S} for f , $h \equiv 0$.

Lemma 3.9 (Integral formulation). *Solutions U of (3.1) may be expressed as*

$$(3.35) \quad U(x, t) = \mathcal{S}(t)U_0 + \int_0^t \mathcal{S}(t-s)f(\cdot, s) + \Gamma U(0, \tilde{x}, t)$$

where $U(x, 0) = U_0(x)$,

$$(3.36) \quad \Gamma U(0, \tilde{x}, t) := \int_0^t \int_{\mathbb{R}^{d-1}} \left(\sum_j G_{y_j} B^{j1} + GA^1 \right) (x, t-s; 0, \tilde{y}) U(0, \tilde{y}, s) d\tilde{y} ds,$$

and $G(\cdot, t; y) = \mathcal{S}(t)\delta_y(\cdot)$ is the Green function of $\partial_t - L$.

Proof. Integrating on \mathbb{R}_+^d the linearized equations

$$(\partial_s - L_y)U = f$$

against $G(x, t - s; y)$ and using the fact that by duality

$$(\partial_s - L_y)^* G^*(x, t - s; y) \equiv 0,$$

we easily obtain the lemma as in the one-dimensional case (see [YZ, NZ]), recalling that

$$\mathcal{S}(t)f = \int_{\mathbb{R}_+^d} G(x, t; y)f(y)dy.$$

□

3.5. Proof of linearized stability.

Proof of Theorem 1.6. Writing the Duhamel formula for the linearized equations

$$(3.37) \quad U(x, t) = \mathcal{S}(t)U_0 + \Gamma h(\tilde{x}, t),$$

with Γh defined in (3.25), where $U(x, 0) = U_0(x)$ and $U(0, \tilde{x}, t) = h(\tilde{x}, t)$, and applying estimates on low- and high-frequency operators $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$, we obtain

$$(3.38) \quad \begin{aligned} |U(t)|_{L^2} &\leq |\mathcal{S}_1(t)U_0|_{L^2} + |\mathcal{S}_2(t)U_0|_{L^2} + |\Gamma h(t)|_{L^2} \\ &\leq C(1+t)^{-\frac{d-1}{4}}|U_0|_{L^1} + Ce^{-\eta t}|U_0|_{H^3} + CE_0(1+t)^{-(d-1)/4} \\ &\leq C(1+t)^{-\frac{d-1}{4}}(|U_0|_{L^1 \cap H^3} + E_0) \end{aligned}$$

and

$$(3.39) \quad \begin{aligned} |U(t)|_{L^\infty} &\leq |\mathcal{S}_1(t)U_0|_{L^\infty} + |\mathcal{S}_2(t)U_0|_{L^\infty} + |\Gamma h(t)|_{L^\infty} \\ &\leq C(1+t)^{-\frac{d}{2}}|U_0|_{L^1} + C|\mathcal{S}_2(t)U_0|_{H^{[(d-1)/2]+2}} + CE_0(1+t)^{-d/2} \\ &\leq C(1+t)^{-\frac{d}{2}}|U_0|_{L^1} + Ce^{-\eta t}|U_0|_{H^{[(d-1)/2]+5}} + CE_0(1+t)^{-d/2} \\ &\leq C(1+t)^{-\frac{d}{2}}(|U_0|_{L^1 \cap H^{[(d-1)/2]+5}} + E_0). \end{aligned}$$

These prove the bounds as stated in the theorem for $p = 2$ and $p = \infty$. For $2 < p < \infty$, we use the interpolation inequality between L^2 and L^∞ . □

4. NONLINEAR STABILITY

4.1. Auxiliary energy estimates. For the analysis of nonlinear stability, we need the following energy estimate adapted from [MaZ4, NZ, Z4]. Define the nonlinear perturbation variables $U = (u, v)$ by

$$(4.1) \quad U(x, t) := \tilde{U}(x, t) - \bar{U}(x_1).$$

Proposition 4.1. *Under the hypotheses of Theorem 1.7, let $U_0 \in H^s$ and $U = (u, v)^T$ be a solution of (1.2) and (4.1). Suppose that, for $0 \leq t \leq T$, the $W_x^{2,\infty}$ norm of the solution U remains bounded by a sufficiently small constant $\zeta > 0$. Then*

$$(4.2) \quad |U(t)|_{H^s}^2 \leq Ce^{-\theta t}|U_0|_{H^s}^2 + C \int_0^t e^{-\theta(t-\tau)} (|U(\tau)|_{L^2}^2 + |\mathcal{B}_h(\tau)|^2) d\tau$$

for all $0 \leq t \leq T$, where the boundary term \mathcal{B}_h is defined as in Theorem 1.7.

Proof. Observe that a straightforward calculation shows that $|U|_{H^r} \sim |W|_{H^r}$,

$$(4.3) \quad W = \tilde{W} - \bar{W} := W(\tilde{U}) - W(\bar{U}),$$

for $0 \leq r \leq s$, provided $|U|_{W^{2,\infty}}$ remains bounded, hence it is sufficient to prove a corresponding bound in the special variable W . We first carry out a complete proof in the more straightforward case with conditions (A1)-(A3) replaced by the following global versions, indicating afterward by a few brief remarks the changes needed to carry out the proof in the general case.

(A1') $\tilde{A}^j(\tilde{W})$, \tilde{A}^0 , \tilde{A}_{11}^1 are symmetric, $\tilde{A}^0 \geq \theta_0 > 0$,

(A2') Same as (A2),

(A3') $\tilde{W} = \begin{pmatrix} \tilde{w}^I \\ \tilde{w}^{II} \end{pmatrix}$, $\tilde{B}^{jk} = \tilde{B}^{kj} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b}^{jk} \end{pmatrix}$, $\sum \xi_j \xi_k \tilde{b}^{jk} \geq \theta |\xi|^2$, and $\tilde{G} \equiv 0$.

Substituting (4.3) into (1.4), we obtain the quasilinear perturbation equation

$$(4.4) \quad A^0 W_t + \sum_j A^j W_{x_j} = \sum_{jk} (B^{jk} W_{x_k})_{x_j} + M_1 \bar{W}_{x_1} + \sum_j (M_2^j \bar{W}_{x_1})_{x_j}$$

where $A^0 := A^0(W + \bar{W})$ is symmetric positive definite, $A^j := A^j(W + \bar{W})$ are symmetric,

$$M_1 = A^1(W + \bar{W}) - A^1(\bar{W}) = \left(\int_0^1 dA^1(\bar{W} + \theta W) d\theta \right) W,$$

$$M_2^j = B^{j1}(W + \bar{W}) - B^{j1}(\bar{W}) = \begin{pmatrix} 0 & 0 \\ 0 & (\int_0^1 db^{j1}(\bar{W} + \theta W) d\theta) W \end{pmatrix}.$$

As shown in [MaZ4], we have bounds

$$(4.5) \quad |A^0| \leq C, \quad |A_t^0| \leq C|W_t| \leq C(|W_x| + |w_{xx}^{II}|) \leq C\zeta,$$

$$(4.6) \quad |\partial_x A^0| + |\partial_x^2 A^0| \leq C \left(\sum_{k=1}^2 |\partial_x^k W| + |\bar{W}_{x_1}| \right) \leq C(\zeta + |\bar{W}_{x_1}|).$$

We have the same bounds for A^j , B^{jk} , and also due to the form of M_1, M_2 ,

$$(4.7) \quad |M_1|, |M_2| \leq C(\zeta + |\bar{W}_{x_1}|)|W|.$$

Note that thanks to Lemma 1.3 we have the bound on the profile: $|\bar{W}_{x_1}| \leq Ce^{-\theta|x_1|}$, as $x_1 \rightarrow +\infty$.

The following results assert that hyperbolic effects can compensate for degenerate viscosity B , as revealed by the existence of a compensating matrix K .

Lemma 4.2 ([KSh]). *Assuming (A1'), condition (A2') is equivalent to the following:*

(K1) *There exist smooth skew-symmetric "compensating matrices" $K(\xi)$, homogeneous degree one in ξ , such that*

$$(4.8) \quad \Re e \left(\sum_{j,k} \xi_j \xi_k B^{jk} - K(\xi)(A^0)^{-1} \sum_k \xi_k A^k \right) (W_+) \geq \theta_2 |\xi|^2 > 0$$

for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

Define α by the ODE

$$(4.9) \quad \alpha_{x_1} = -\text{sign}(A_{11}^1)c_*|\bar{W}_{x_1}|\alpha, \quad \alpha(0) = 1$$

where $c_* > 0$ is a large constant to be chosen later. Note that we have

$$(4.10) \quad (\alpha_{x_1}/\alpha)A_{11}^1 \leq -c_*\theta_1|\bar{W}_{x_1}| =: -\omega(x_1)$$

and

$$(4.11) \quad |\alpha_{x_1}/\alpha| \leq c_*|\bar{W}_{x_1}| = \theta_1^{-1}\omega(x_1).$$

In what follows, we shall use $\langle \cdot, \cdot \rangle$ as the α -weighted L^2 inner product defined as

$$\langle f, g \rangle = \langle \alpha f, g \rangle_{L^2(\mathbb{R}_+^d)}$$

and

$$\|f\|_s = \sum_{i=0}^s \sum_{|\alpha|=i} \left\langle \partial_x^\alpha f, \partial_x^\alpha f \right\rangle^{1/2}$$

as the norm in weighted H^s space. Note that for any symmetric operator S ,

$$\begin{aligned} \langle Sf_{x_j}, f \rangle &= -\frac{1}{2} \langle S_{x_j} f, f \rangle, \quad j \neq 1 \\ \langle Sf_{x_1}, f \rangle &= -\frac{1}{2} \langle (S_{x_1} + (\alpha_{x_1}/\alpha)S) f, f \rangle - \frac{1}{2} \langle Sf, f \rangle_0, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_0$ denotes the integration on $\mathbb{R}_0^d := \{x_1 = 0\} \times \mathbb{R}^{d-1}$. Also we define

$$\|f\|_{0,s} = \|f\|_{H^s(\mathbb{R}_0^d)} = \sum_{i=0}^s \sum_{|\alpha|=i} \left\langle \partial_x^\alpha f, \partial_x^\alpha f \right\rangle_0^{1/2}.$$

Note that in what follows, we shall pay attention to keeping track of c_* . For constants independent of c_* , we simply write them as C . Also, for simplicity, the sum symbol will sometimes be dropped where it is no confusion. We write $\|f_x\| = \sum_j \|f_{x_j}\|$ and $\|\partial_x^k f\| = \sum_{|\alpha|=k} \|\partial_x^\alpha f\|$.

4.1.1. *Zeroth order “Friedrichs-type” estimate.* First, by integration by parts and estimates (4.5), (4.6), and then (4.10), we obtain for $j \neq 1$,

$$-\langle A^j W_{x_j}, W \rangle = \frac{1}{2} \langle A_{x_j}^j W, W \rangle \leq C \langle (\zeta + |\bar{W}_{x_1}|) w^I, w^I \rangle + C \|w^{II}\|_0^2$$

and for $j = 1$,

$$\begin{aligned} -\langle A^1 W_{x_1}, W \rangle &= \frac{1}{2} \langle (A_{x_1}^1 + (\alpha_{x_1}/\alpha)A^1) W, W \rangle + \frac{1}{2} \langle A^1 W, W \rangle_0 \\ &\leq \frac{1}{2} \langle (\alpha_{x_1}/\alpha)A_{11}^1 w^I, w^I \rangle + C \langle (\zeta + |\bar{W}_{x_1}|) |W| + \omega(x_1) |w^{II}|, |W| \rangle + J_b^0 \\ &\leq -\frac{1}{2} \langle \omega(x) w^I, w^I \rangle + C \langle (\zeta + |\bar{W}_{x_1}|) w^I, w^I \rangle + C(c_*) \|w^{II}\|_0^2 + J_b^0, \end{aligned}$$

where J_b^0 denotes the boundary term $\frac{1}{2}\langle A^1 W, W \rangle_0$. The term $\langle |\bar{W}_{x_1}| w^I, w^I \rangle$ may be easily absorbed into the first term of the right-hand side, since for c_* sufficiently large,

$$(4.12) \quad \langle |\bar{W}_{x_1}| w^I, w^I \rangle \leq (c_* \theta_1)^{-1} \langle \omega(x_1) w^I, w^I \rangle \leq \frac{1}{4C} \langle \omega(x_1) w^I, w^I \rangle.$$

Also, integration by parts yields

$$\begin{aligned} \langle (B^{jk} W_{x_k})_{x_j}, W \rangle &= -\langle B^{jk} W_{x_k}, W_{x_j} \rangle - \langle (\alpha_{x_1}/\alpha) B^{1k} W_{x_k}, W \rangle - \langle B^{1k} W_{x_k}, W \rangle_0 \\ &\leq -\theta \|w_x^{II}\|_0^2 + C \langle \omega(x_1) w_x^{II}, w^{II} \rangle - \langle b^{1k} w_{x_k}^{II}, w^{II} \rangle_0 \\ &\leq -\theta \|w_x^{II}\|_0^2 + C(c_*) \|w^{II}\|_0^2 - \langle b^{1k} w_{x_k}^{II}, w^{II} \rangle_0. \end{aligned}$$

where we used the fact that $B^{jk} W_x \cdot W = b^{jk} w_x^{II} \cdot w^{II}$, noting that B has block-diagonal form with the first block identical to zero. Similarly, recalling that $M_2^j = B^{j1}(W + \bar{W}) - B^{j1}(\bar{W})$, we have

$$\begin{aligned} \langle (M_2^j \bar{W}_{x_1})_{x_j}, W \rangle &= -\langle M_2^j \bar{W}_{x_1}, W_{x_j} \rangle - \langle (\alpha_{x_1}/\alpha) M_2^1 \bar{W}_{x_1}, W \rangle - \langle M_2^1 \bar{W}_{x_1}, W \rangle_0 \\ &\leq C \langle |\bar{W}_{x_1}| |W|, |w_x^{II}| \rangle + C \langle \omega(x_1) |W|, w^{II} \rangle - \langle m_2^1 \bar{W}_{x_1}, w^{II} \rangle_0 \\ &\leq \xi \|w_x^{II}\|_0^2 + C \left(\epsilon \langle \omega(x_1) w^I, w^I \rangle + C(c_*) \|w^{II}\|_0^2 \right) - \langle m_2^1 \bar{W}_{x_1}, w^{II} \rangle_0 \end{aligned}$$

for any small ξ, ϵ . Note that C is independent of c_* . Therefore, for $\xi = \theta/2$ and c_* sufficiently large, combining all above estimates, we obtain

$$\begin{aligned} (4.13) \quad \frac{1}{2} \frac{d}{dt} \langle A^0 W, W \rangle &= \langle A^0 W_t, W \rangle + \frac{1}{2} \langle A_t^0 W, W \rangle \\ &= \langle -A^j W_{x_j} + (B^{jk} W_{x_k})_{x_j} + M_1 \bar{W}_{x_1} + (M_2^j \bar{W}_{x_1})_{x_j}, W \rangle + \frac{1}{2} \langle A_t^0 W, W \rangle \\ &\leq -\frac{1}{4} [\langle \omega(x_1) w^I, w^I \rangle + \theta \|w_x^{II}\|_0^2] + C\zeta \|w^I\|_0^2 + C(c_*) \|w^{II}\|_0^2 + I_b^0 \end{aligned}$$

where the boundary term

$$(4.14) \quad I_b^0 := \frac{1}{2} \langle A^1 W, W \rangle_0 - \langle b^{1k} w_{x_k}^{II}, w^{II} \rangle_0 - \langle m_2^1 \bar{W}_{x_1}, w^{II} \rangle_0$$

which, in the outflow case (thanks to the negative definiteness of A_{11}^1), is estimated as

$$(4.15) \quad I_b^0 \leq -\frac{\theta_1}{2} \|w^I\|_{0,0}^2 + C(\|w^{II}\|_{0,0}^2 + \|w_x^{II}\|_{0,0} \|w^{II}\|_{0,0}),$$

and similarly in the inflow case, estimated as

$$(4.16) \quad I_b^0 \leq C(\|W\|_{0,0}^2 + \|w_x^{II}\|_{0,0} \|w^{II}\|_{0,0}).$$

Here we recall that $\|\cdot\|_{0,s} := \|\cdot\|_{H^s(\mathbb{R}_0^d)}$.

4.1.2. *First order “Friedrichs-type” estimate.* Similarly as above, we need the following key estimate, computing by the use of integration by parts, (4.12), and c_* being sufficiently large,

$$\begin{aligned}
(4.17) \quad & - \sum_j \langle W_{x_i}, A^j W_{x_i x_j} \rangle \\
&= \frac{1}{2} \sum_j \langle W_{x_i}, A_{x_j}^j W_{x_i} \rangle + \frac{1}{2} \langle W_{x_i}, (\alpha_{x_1}/\alpha) A^1 W_{x_i} \rangle + \frac{1}{2} \langle W_{x_i}, A^1 W_{x_i} \rangle_0 \\
&\leq -\frac{1}{4} \langle \omega(x_1) w_x^I, w_x^I \rangle + C\zeta \|w_x^I\|_0^2 + Cc_*^2 \|w_x^{II}\|_0^2 + \frac{1}{2} \langle W_{x_i}, A^1 W_{x_i} \rangle_0.
\end{aligned}$$

We deal with the boundary term later. Now let us compute

$$(4.18) \quad \frac{1}{2} \frac{d}{dt} \langle A^0 W_{x_i}, W_{x_i} \rangle = \langle W_{x_i}, (A^0 W_t)_{x_i} \rangle - \langle W_{x_i}, A_{x_i}^0 W_t \rangle + \frac{1}{2} \langle A_t^0 W_{x_i}, W_{x_i} \rangle.$$

We control each term in turn. By (4.5) and (4.6), we first have

$$\langle A_t^0 W_{x_i}, W_{x_i} \rangle \leq C\zeta \|W_x\|_0^2$$

and by multiplying $(A^0)^{-1}$ into (4.4),

$$\begin{aligned}
|\langle W_{x_i}, A_{x_i}^0 W_t \rangle| &\leq C \langle (\zeta + |\bar{W}_{x_1}|) |W_x|, (|W_x| + |w_{xx}^{II}| + |W|) \rangle \\
&\leq \xi \|w_{xx}^{II}\|_0^2 + C \langle (\zeta + |\bar{W}_{x_1}|) w_x^I, w_x^I \rangle + C \langle (\zeta + |\bar{W}_{x_1}|) w^I, w^I \rangle + C \|w^{II}\|_1^2,
\end{aligned}$$

where the term $\langle |\bar{W}_{x_1}| w_x^I, w_x^I \rangle$ may be treated in the same way as was $\langle |\bar{W}_{x_1}| w^I, w^I \rangle$ in (4.12). Using (4.4), we write the first term in the right-hand side of (4.18) as

$$\begin{aligned}
\langle W_{x_i}, (A^0 W_t)_{x_i} \rangle &= \langle W_{x_i}, [-A^j W_{x_j} + (B^{jk} W_{x_k})_{x_j} + M_1 \bar{W}_{x_1} + (M_2^j \bar{W}_{x_1})_{x_j}]_{x_i} \rangle \\
&= -\langle W_{x_i}, A^j W_{x_i x_j} \rangle + \langle W_{x_i}, -A_{x_i}^j W_{x_j} + (M_1 \bar{W}_{x_1})_{x_i} \rangle \\
&\quad - \langle W_{x_i x_j}, [(B^{jk} W_{x_k})_{x_i} + (M_2^j \bar{W}_{x_1})_{x_i}] \rangle \\
&\quad - \langle (\alpha_{x_1}/\alpha) W_{x_i}, [(B^{1k} W_{x_k})_{x_i} + (M_2^1 \bar{W}_{x_1})_{x_i}] \rangle \\
&\quad - \langle W_{x_i}, [(B^{1k} W_{x_k})_{x_i} + (M_2^1 \bar{W}_{x_1})_{x_i}] \rangle_0 \\
&\leq -\frac{1}{4} \left[\langle \omega(x_1) w_x^I, w_x^I \rangle + \theta \|w_{xx}^{II}\|_0^2 \right] \\
&\quad + C \left[\zeta \|w^I\|_1^2 + C(c_*) \|w_x^{II}\|_0^2 + \langle |\bar{W}_{x_1}| w^I, w^I \rangle \right] + I_b^1
\end{aligned}$$

where I_b^1 denotes the boundary terms

$$\begin{aligned}
(4.19) \quad I_b^1 &:= \frac{1}{2} \langle W_{x_i}, A^1 W_{x_i} \rangle_0 - \langle W_{x_i}, [(B^{1k} W_{x_k})_{x_i} + (M_2^1 \bar{W}_{x_1})_{x_i}] \rangle_0 \\
&= \frac{1}{2} \langle W_{x_i}, A^1 W_{x_i} \rangle_0 - \langle w_{x_i}^{II}, [(b^{1k} w_{x_k}^{II})_{x_i} + (m_2^1 \bar{W}_{x_1})_{x_i}] \rangle_0,
\end{aligned}$$

and we have used (A3) for each fixed i and $\xi_j = (W_{x_i})_{x_j}$ to get

$$(4.20) \quad \sum_{jk} \langle W_{x_i x_j}, B^{jk} W_{x_k x_i} \rangle \geq \theta \sum_j \|W_{x_i x_j}\|_0^2,$$

and estimates (4.17),(4.12) for w^I, w_x^I , and Young's inequality to obtain:

$$\begin{aligned} \langle W_x, -A_x^j W_x + (M_1 \bar{W}_{x_1})_x \rangle &\leq C \langle (\zeta + |\bar{W}_{x_1}|) |W_x|, |W_x| + |W| \rangle. \\ -\langle W_{xx} + (\alpha_{x_1}/\alpha) W_x, (B^{jk} W_x)_x \rangle &\leq \\ &\quad -\theta \|w_{xx}^{II}\|_0^2 + C \langle |w_{xx}^{II}| + \omega(x_1) |w_x^{II}|, (\zeta + |\bar{W}_{x_1}|) |w_x^{II}| \rangle \\ -\langle W_{xx} + (\alpha_{x_1}/\alpha) W_x, (M_2^j \bar{W}_{x_1})_x \rangle &\leq \\ &\quad C \langle |w_{xx}^{II}| + \omega(x_1) |w_x^{II}|, (\zeta + |\bar{W}_{x_1}|) (|W_x| + |W|) \rangle. \end{aligned}$$

Putting these estimates together into (4.18), we have obtained

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \langle A^0 W_x, W_x \rangle + \frac{1}{4} \theta \|w_{xx}^{II}\|_0^2 + \frac{1}{4} \langle \omega(x_1) w_x^I, w_x^I \rangle \\ (4.21) \quad \leq C \left[\zeta \|w^I\|_1^2 + \langle |\bar{W}_{x_1}| w^I, w^I \rangle + C(c_*) \|w^{II}\|_1^2 \right] + I_b^1. \end{aligned}$$

Let us now treat the boundary term. First observe that using the parabolic equations, noting that A^0 is the diagonal-block form, we can estimate

$$(4.22) \quad (b^{jk} w_{x_k}^{II})_{x_j}(0, \tilde{x}, t) \leq C \left(|w_t^{II}| + |W_{x_j}| + |W| \right) (0, \tilde{x}, t)$$

and thus for $i \neq 1$

$$\begin{aligned} \langle w_{x_i}^{II}, [(b^{1k} w_{x_k}^{II})_{x_i} + (m_2^1 \bar{W}_{x_1})_{x_i}]_0 \rangle \\ (4.23) \quad \leq \int_{\mathbb{R}_0^d} |w_{x_i x_i}^{II}| (|W| + |w_{x_k}^{II}|) \\ \leq C \int_{\mathbb{R}_0^d} (|W|^2 + |w_x^{II}|^2 + |w_{\tilde{x}\tilde{x}}^{II}|^2) \end{aligned}$$

and for $i = 1$, using $b^{1k} = b^{k1}$, (4.22), and recalling here that we always use the sum convention,

$$\begin{aligned} \sum_k (b^{1k} w_{x_k}^{II})_{x_1} &= \frac{1}{2} \left((b^{1k} w_{x_k}^{II})_{x_1} + (b^{j1} w_{x_1}^{II})_{x_j} + b_{x_1}^{1k} w_{x_k}^{II} - b_{x_j}^{j1} w_{x_1}^{II} \right) \\ (4.24) \quad &= \frac{1}{2} \left((b^{jk} w_{x_k}^{II})_{x_j} + b_{x_1}^{1k} w_{x_k}^{II} - b_{x_j}^{j1} w_{x_1}^{II} - \sum_{j \neq 1; k \neq 1} (b^{jk} w_{x_k}^{II})_{x_j} \right) \\ &\leq C (|w_t^{II}| + |W| + |W_{x_j}| + |w_{\tilde{x}\tilde{x}}^{II}|). \end{aligned}$$

Therefore

$$\begin{aligned} \langle w_{x_1}^{II}, [(b^{1k} w_{x_k}^{II})_{x_1} + (m_2^1 \bar{W}_{x_1})_{x_1}]_0 \rangle \\ \leq \epsilon \int_{\mathbb{R}_0^d} |w_x^I|^2 + C \int_{\mathbb{R}_0^d} (|w_t^{II}|^2 + |W|^2 + |w_x^{II}|^2 + |w_{\tilde{x}\tilde{x}}^{II}|^2) \end{aligned}$$

For the first term in I_b^1 , we consider each inflow/outflow case separately. For the outflow case, since $A_{11}^1 \leq -\theta_1 < 0$, we get

$$A^1 W_x \cdot W_x \leq -\frac{\theta_1}{2} |w_x^I|^2 + C |w_x^{II}|^2.$$

Therefore

$$(4.25) \quad I_b^1 \leq -\frac{\theta_1}{2} \int_{\mathbb{R}_0^d} |w_x^I|^2 + \int_{\mathbb{R}_0^d} \left(|W|^2 + |w_x^{II}|^2 + |w_t^{II}|^2 + |w_{\tilde{x}\tilde{x}}^{II}|^2 \right).$$

Meanwhile, for the inflow case, since $A_{11}^1 \geq \theta_1 > 0$, we have

$$|A^1 W_x \cdot W_x| \leq C |W_x|^2.$$

In this case, the invertibility of A_{11}^1 allows us to use the hyperbolic equation to derive

$$|w_{x_1}^I| \leq C(|w_t^I| + |w_x^{II}| + |w_{\tilde{x}}^I|).$$

Therefore we get

$$(4.26) \quad I_b^1 \leq \int_{\mathbb{R}_0^d} \left(|W|^2 + |W_t|^2 + |w_{\tilde{x}}^I|^2 + |w_x^{II}|^2 + |w_{\tilde{x}\tilde{x}}^{II}|^2 \right).$$

Now apply the standard Sobolev inequality

$$(4.27) \quad |w(0)|^2 \leq C \|w\|_{L^2(\mathbb{R})} (\|w_x\|_{L^2(\mathbb{R})} + \|w\|_{L^2(\mathbb{R})})$$

to control the term $|w_{x_1}^{II}(0)|^2$ in I_b^1 in both cases. We get

$$(4.28) \quad \int_{\mathbb{R}_0^d} |w_{x_1}^{II}|^2 \leq \epsilon' \|w_{xx}^{II}\|_0^2 + C \|w_x^{II}\|_0^2.$$

Using this with $\epsilon' = \theta/8$, (4.19), and (4.25), the estimate (4.21) reads

$$(4.29) \quad \begin{aligned} \frac{d}{dt} \langle A^0 W_x, W_x \rangle + \|w_{xx}^{II}\|_0^2 + \langle \omega(x_1) w_x^I, w_x^I \rangle \\ \leq C \left(\zeta \|w^I\|_1^2 + \langle |\bar{W}_{x_1}| w^I, w^I \rangle + C(c_*) \|w^{II}\|_1^2 \right) + I_b^1 \end{aligned}$$

where the (new) boundary term I_b^1 satisfies

$$(4.30) \quad I_b^1 \leq -\frac{\theta_1}{2} \int_{\mathbb{R}_0^d} |w_x^I|^2 + C \int_{\mathbb{R}_0^d} \left(|W|^2 + |w_{\tilde{x}}^{II}|^2 + |w_t^{II}|^2 + |w_{\tilde{x}\tilde{x}}^{II}|^2 \right)$$

for the outflow case, and

$$(4.31) \quad I_b^1 \leq \int_{\mathbb{R}_0^d} \left(|W|^2 + |W_t|^2 + |W_{\tilde{x}}|^2 + |w_{\tilde{x}\tilde{x}}^{II}|^2 \right)$$

for the inflow case.

4.1.3. Higher order “Friedrichs-type” estimate. For any fixed multi-index $\alpha = (\alpha_{x_1}, \dots, \alpha_{x_d})$, $\alpha_1 = 0, 1$, $|\alpha| = k = 2, \dots, s$, by computing $\frac{d}{dt} \langle A^0 \partial_x^\alpha W, \partial_x^\alpha W \rangle$ and following the same spirit as the above subsection, we easily obtain

$$(4.32) \quad \begin{aligned} \frac{d}{dt} \langle A^0 \partial_x^\alpha W, \partial_x^\alpha W \rangle + \theta \|\partial_x^{\alpha+1} w^{II}\|_0^2 + \langle \omega(x_1) \partial_x^\alpha w^I, \partial_x^\alpha w^I \rangle \\ \leq C \left(C(c_*) \|w^{II}\|_k^2 + \zeta \|w^I\|_k^2 + \sum_{i=1}^{k-1} \langle |\bar{W}_{x_1}| \partial_x^i w^I, \partial_x^i w^I \rangle \right) + I_b^\alpha \end{aligned}$$

where

$$\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}, \quad \partial_x^{\alpha+1} := \sum_j \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} \partial_{x_j}, \quad \partial_x^i = \sum_{|\beta|=i} \partial_{x_1}^{\beta_1} \cdots \partial_{x_d}^{\beta_d}$$

and the boundary term I_b^α satisfies

$$(4.33) \quad I_b^\alpha \leq -\frac{\theta_1}{2} \int_{\mathbb{R}_0^d} |\partial_x^\alpha w^I|^2 + C \int_{\mathbb{R}_0^d} \left(\sum_{i=1}^{[(k+1)/2]} |\partial_t^i w^{II}|^2 + \sum_{i=0}^{k-1} |\partial_x^i w^I|^2 + \sum_{i=0}^k |\partial_x^i w^{II}|^2 \right)$$

for the outflow case, and

$$(4.34) \quad I_b^\alpha \leq \int_{\mathbb{R}_0^d} \left(\sum_{i=0}^k |\partial_t^i w^I|^2 + \sum_{i=1}^{[(k+1)/2]} |\partial_t^i w^{II}|^2 + \sum_{i=0}^k |\partial_x^i w^I|^2 \right)$$

for the inflow case.

Now for α with $\alpha_1 = 2, \dots, s$ we observe that the estimate (4.32) still holds. Indeed, using integration by parts and computing $\frac{d}{dt} \langle A^0 \partial_x^\alpha W, \partial_x^\alpha W \rangle$ as above leaves the boundary terms as

$$(4.35) \quad I_b^\alpha := \frac{1}{2} \langle \partial_x^\alpha W, A^1 \partial_x^\alpha W \rangle_0 - \langle \partial_x^\alpha w^{II}, \partial_x^\alpha [(b^{1k} w_{x_k}^{II}) + (m_2^1 \bar{W}_{x_1})] \rangle_0.$$

Then we can use the parabolic equations to solve

$$w_{x_1 x_1}^{II} = (b^{11})^{-1} \left(A_2^0 w_t^{II} + A_2^j W_{x_j} - (b^{jk} w_{x_k}^{II})_{x_j} - b_{x_1}^{11} w_{x_1}^{II} - M_1 \bar{W}_{x_1} - (m_2^j \bar{W}_{x_1})_{x_j} \right).$$

Using this we can reduce the order of derivative with respect to x_1 in ∂_x^α to one, with the same spirit as (4.23) and (4.24). Finally we use the Sobolev embedding similar to (4.28) to obtain the estimate for the normal derivative ∂_{x_1} , and get the estimate for I_b^α as claimed in (4.33) and (4.34).

We recall next the following Kawashima-type estimate, presented in [Z3], to bound the term $\|w^I\|_k^2$ appearing on the left hand side of (4.32).

4.1.4. “*Kawashima-type*” estimate. Let $K(\xi)$ be the skew-symmetry in (4.8). Using Plancherel’s identity and the equations (4.4), we compute

$$(4.36) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \langle K(\partial_x) \partial_x^r W, \partial_x^r W \rangle &= \frac{1}{2} \frac{d}{dt} \langle iK(\xi) (i\xi)^r \hat{W}, (i\xi)^r \hat{W} \rangle \\ &= \langle iK(\xi) (i\xi)^r \hat{W}, (i\xi)^r \hat{W}_t \rangle \\ &= \langle (i\xi)^r \hat{W}, -K(\xi) (A_+^0)^{-1} \sum_j \xi_j A_+^j (i\xi)^r \hat{W} \rangle \\ &\quad + \langle iK(\xi) (i\xi)^r \hat{W}, (i\xi)^r \hat{H} \rangle, \end{aligned}$$

where

$$(4.37) \quad \begin{aligned} H &:= \sum_j \left((A_+^0)^{-1} A_+^j - (A^0)^{-1} A^j \right) W_{x_j} \\ &\quad + (A^0)^{-1} \left(\sum_{jk} (B^{jk} W_{x_k})_{x_j} + M_1 \bar{W}_{x_1} + \sum_j (M_2^j \bar{W}_{x_1})_{x_j} \right). \end{aligned}$$

By using the fact that $|(A_+^0)^{-1}A_+^j - (A^0)^{-1}A^j| = \mathcal{O}(\zeta + |\bar{W}_{x_1}|)$, we can easily obtain

$$\|\partial_x^r H\|_0^2 \leq C\|w^{II}\|_{r+2}^2 + C \sum_{k=0}^{r+1} \langle (\zeta + |\bar{W}_{x_1}|) \partial_x^k w^I, \partial_x^k w^I \rangle.$$

Meanwhile, applying (4.8) into the first term of the last line in (4.36), we get

$$\begin{aligned} & \langle (i\xi)^r \hat{W}, -K(\xi)(A_+^0)^{-1} \sum_j \xi_j A_+^j (i\xi)^r \hat{W} \rangle \\ & \geq \theta \|\xi|^{r+1} \hat{W}\|_0^2 - C \|\xi|^{r+1} \hat{w}^{II}\|_0^2 \\ & = \theta \|\partial_x^{r+1} w^I\|_0^2 - C \|\partial_x^{r+1} w^{II}\|_0^2. \end{aligned}$$

Putting these estimates together into (4.36), we have obtained the high order “Kawashima-type” estimate:

$$\begin{aligned} (4.38) \quad \frac{d}{dt} \langle K(\partial_x) \partial_x^r W, \partial_x^r W \rangle & \leq -\theta \|\partial_x^{r+1} w^I\|_0^2 + C \|w^{II}\|_{r+2}^2 \\ & + C \sum_{i=0}^{r+1} \langle (\zeta + |\bar{W}_{x_1}|) \partial_x^i w^I, \partial_x^i w^I \rangle \end{aligned}$$

4.1.5. *Final estimates.* We are ready to conclude our result. First combining the estimate (4.29) with (4.13), we easily obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\langle A^0 W_x, W_x \rangle + M \langle A^0 W, W \rangle \right) \\ & \leq - \left(\frac{\theta}{8} \|w_{xx}^{II}\|_0^2 + \frac{1}{4} \langle \omega(x_1) w_x^I, w_x^I \rangle \right) \\ & \quad + C \left(\zeta \|w^I\|_1^2 + \langle |\bar{W}_{x_1}| w^I, w^I \rangle + C(c_*) \|w^{II}\|_1^2 \right) + I_b^1 \\ & \quad - \frac{M}{4} \left(\langle \omega(x_1) w^I, w^I \rangle + \theta \|w_x^{II}\|_0^2 \right) + CM\zeta \|w^I\|_0^2 + MC(c_*) \|w^{II}\|_0^2 + MI_b^0 \end{aligned}$$

By choosing M sufficiently large such that $M\theta \gg C(c_*)$, and noting that $c_*\theta_1|\bar{W}_{x_1}| \leq \omega(x_1)$, we get

$$\begin{aligned} (4.39) \quad & \frac{1}{2} \frac{d}{dt} \left(\langle A^0 W_x, W_x \rangle + M \langle A^0 W, W \rangle \right) \\ & \leq - \left(\theta \|w^{II}\|_2^2 + \langle \omega(x_1) w^I, w^I \rangle + \langle \omega(x_1) w_x^I, w_x^I \rangle \right) \\ & \quad + C \left(\zeta \|w^I\|_1^2 + C(c_*) \|w^{II}\|_0^2 \right) + I_b^1 + MI_b^0. \end{aligned}$$

We shall treat the boundary terms later. Now we use the estimate (4.38) (for $r = 0$) to absorb the term $\|\partial_x w^I\|_0$ into the left hand side. Indeed, fixing c_* large as above, adding (4.39) with (4.38) times ϵ , and choosing ϵ, ζ sufficiently small such that $\epsilon C(c_*) \ll \theta, \epsilon \ll 1$ and $\zeta \ll \epsilon\theta_2$, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\langle A^0 W_x, W_x \rangle + M \langle A^0 W, W \rangle + \epsilon \langle K W_x, W \rangle \right) \\
& \leq - \left(\theta \|w^{II}\|_2^2 + \langle \omega(x_1) w^I, w^I \rangle + \langle \omega(x_1) w_x^I, w_x^I \rangle \right) \\
& \quad + C \left(\zeta \|w^I\|_1^2 + C(c_*) \|w^{II}\|_0^2 \right) - \frac{\theta_2 \epsilon}{2} \|w_x^I\|_0^2 \\
& \quad + C \epsilon \left(\|w^{II}\|_2^2 + \zeta \|w^I\|_0^2 + \langle \omega(x_1) w^I, w^I \rangle + \langle \omega(x_1) w_x^I, w_x^I \rangle \right) + I_b^1 + M I_b^0 \\
& \leq - \frac{1}{2} \left(\theta \|w^{II}\|_2^2 + \theta_2 \epsilon \|w_x^I\|_0^2 \right) + C(c_*) \|W\|_0^2 + I_b
\end{aligned}$$

where $I_b := I_b^1 + M I_b^0$.

In view of boundary terms I_b^0 and I_b^1 , we treat the term I_b in each inflow/outflow case separately. Recall the inequality (4.28), $\|w_{x_1}^{II}\|_{0,0} \leq C \|w^{II}\|_2$. Thus, using this, for the inflow case we have

$$(4.40) \quad I_b^0 \leq C(\|W\|_{0,0}^2 + \|w_x^{II}\|_{0,0} \|w^{II}\|_{0,0}) \leq C(\|W\|_{0,0}^2 + \|w_{\tilde{x}}^{II}\|_{0,0}^2 + \epsilon \|w^{II}\|_2^2)$$

and for the outflow case,

$$\begin{aligned}
(4.41) \quad I_b^0 & \leq -\frac{\theta_1}{2} \|w^I\|_{0,0}^2 + C(\|w^{II}\|_{0,0}^2 + \|w_x^{II}\|_{0,0} \|w^{II}\|_{0,0}) \\
& \leq -\frac{\theta_1}{2} \|w^I\|_{0,0}^2 + C(\|w^{II}\|_{0,0}^2 + \|w_{\tilde{x}}^{II}\|_{0,0}^2 + \epsilon \|w^{II}\|_2^2).
\end{aligned}$$

Therefore these together with (4.30) and (4.31), using the good estimate of $\|w_{xx}^{II}\|_0^2$, yield

$$(4.42) \quad I_b \leq -\frac{\theta_1}{2} \int_{\mathbb{R}_0^d} (|w^I|^2 + |w_x^I|^2) + C \int_{\mathbb{R}_0^d} (|w^{II}|^2 + |w_{\tilde{x}}^{II}|^2 + |w_t^{II}|^2 + |w_{\tilde{x}\tilde{x}}^{II}|^2)$$

for the outflow case, and

$$(4.43) \quad I_b^1 \leq \int_{\mathbb{R}_0^d} (|W|^2 + |W_t|^2 + |W_{\tilde{x}}|^2 + |w_{\tilde{x}\tilde{x}}^{II}|^2)$$

for the inflow case.

Now by Cauchy-Schwarz's inequality, $|K(\xi)| \leq C|\xi|$, and positive definiteness of A^0 , it is easy to see that

$$(4.44) \quad \mathcal{E} := \langle A^0 W_x, W_x \rangle + M \langle A^0 W, W \rangle + \epsilon \langle K(\partial_x) W, W \rangle \sim \|W\|_{H_\alpha^1}^2 \sim \|W\|_{H^1}^2.$$

The last equivalence is due to the fact that α is bounded above and below away from zero. Thus the above yields

$$\frac{d}{dt} \mathcal{E}(W)(t) \leq -\theta_3 \mathcal{E}(W)(t) + C(c_*) \left(\|W(t)\|_{L^2}^2 + |\mathcal{B}_1(t)|^2 \right),$$

for some positive constant θ_3 , which by the Gronwall inequality implies

$$(4.45) \quad \|W(t)\|_{H^1}^2 \leq C e^{-\theta t} \|W_0\|_{H^1}^2 + C(c_*) \int_0^t e^{-\theta(t-\tau)} \left(\|W(\tau)\|_{L^2}^2 + |\mathcal{B}_1(\tau)|^2 \right) d\tau,$$

where $W(x, 0) = W_0(x)$ and

$$(4.46) \quad |\mathcal{B}_1(\tau)|^2 := \int_{\mathbb{R}_0^d} \left(|W|^2 + |W_t|^2 + |W_{\tilde{x}}|^2 + |w_{\tilde{x}\tilde{x}}^{II}|^2 \right)$$

for the inflow case, and

$$(4.47) \quad |\mathcal{B}_1(\tau)|^2 := \int_{\mathbb{R}_0^d} \left(|w^{II}|^2 + |w_{\tilde{x}}^{II}|^2 + |w_t^{II}|^2 + |w_{\tilde{x}\tilde{x}}^{II}|^2 \right)$$

for the outflow case.

Similarly, by induction, we can derive the same estimates for W in H^s . To do that, let us define

$$\begin{aligned} \mathcal{E}_1(W) &:= \langle A^0 W_x, W_x \rangle + M \langle A^0 W, W \rangle + \epsilon \langle K W_x, W \rangle \\ \mathcal{E}_k(W) &:= \langle A^0 \partial_x^k W, \partial_x^k W \rangle + M \mathcal{E}_{k-1}(W) + \epsilon \langle K \partial_x^k W, \partial_x^{k-1} W \rangle, \quad k \leq s. \end{aligned}$$

Then similarly by the Cauchy-Schwarz inequality, $\mathcal{E}_s(W) \sim \|W\|_{H^s}^2$, and by induction, we obtain

$$\frac{d}{dt} \mathcal{E}_s(W)(t) \leq -\theta_3 \mathcal{E}_s(W)(t) + C(c_*) (\|W(t)\|_{L^2}^2 + |\mathcal{B}_h(t)|^2),$$

for some positive constant θ_3 , which by the Gronwall inequality yields

$$(4.48) \quad \|W(t)\|_{H^s}^2 \leq C e^{-\theta t} \|W_0\|_{H^s}^2 + C(c_*) \int_0^t e^{-\theta(t-\tau)} (\|W(\tau)\|_{L^2}^2 + |\mathcal{B}_h(\tau)|^2) d\tau,$$

where $W(x, 0) = W_0(x)$, and \mathcal{B}_h are defined as in (1.14) and (1.15).

4.1.6. The general case. Following [MaZ4, Z3], the general case that hypotheses (A1)-(A3) hold can easily be covered via following simple observations. First, we may express matrix A in (4.4) as

$$(4.49) \quad A^j(W + \bar{W}) = \hat{A}^j + (\zeta + |\bar{W}_{x_1}|) \begin{pmatrix} 0 & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix},$$

where \hat{A}^j is a symmetric matrix obeying the same derivative bounds as described for A^j , \hat{A}^1 identical to A^1 in the 11 block and obtained in other blocks kl by

$$\begin{aligned} (4.50) \quad A_{kl}^1(W + \bar{W}) &= A_{kl}^1(\bar{W}) + A_{kl}^1(W + \bar{W}) - A_{kl}^1(\bar{W}) \\ &= A_{kl}^1(W_+) + \mathcal{O}(|W_x| + |\bar{W}_{x_1}|) \\ &= A_{kl}^1(W_+) + \mathcal{O}(\zeta + |\bar{W}_{x_1}|) \end{aligned}$$

and meanwhile, \hat{A}^j , $j \neq 1$, obtained by $A^j = A^j(W_+) + \mathcal{O}(\zeta + |\bar{W}_{x_1}|)$, similarly as in (4.50).

Replacing A^j by \hat{A}^j in the k^{th} order Friedrichs-type bounds above, we find that the resulting error terms may be expressed as

$$\langle \partial_x^k \mathcal{O}(\zeta + |\bar{W}_{x_1}|) |W|, |\partial_x^{k+1} w^{II}| \rangle,$$

plus lower order terms, easily absorbed using Young's inequality, and boundary terms

$$\mathcal{O}\left(\sum_{i=0}^k |\partial_x^i w^{II}(0)| |\partial_x^k w^I(0)|\right)$$

resulting from the use of integration by parts as we deal with the 12-block. However these boundary terms were already treated somewhere as before. Hence we can recover the same Friedrichs-type estimates obtained above. Thus we may relax (A1') to (A1).

Next, to relax (A3') to (A3), first we show that the symmetry condition $B^{jk} = B^{kj}$ is not necessary. Indeed, by writing

$$\sum_{jk} (B^{jk} W_{x_k})_{x_j} = \sum_{jk} \left(\frac{1}{2} (B^{jk} + B^{kj}) W_{x_k} \right)_{x_j} + \frac{1}{2} \sum_{jk} (B^{jk} - B^{kj})_{x_j} W_{x_k},$$

we can just replace B^{jk} by $\tilde{B}^{jk} := \frac{1}{2}(B^{jk} + B^{kj})$, satisfying the same (A3'), and thus still obtain the energy estimates as before, with a harmless error term (last term in the above identity). Next notice that the term $g(\tilde{W}_x) - g(\tilde{W}_{x_1})$ in the perturbation equation may be Taylor expanded as

$$\begin{pmatrix} 0 \\ g_1(\tilde{W}_x, \tilde{W}_{x_1}) + g_1(\tilde{W}_{x_1}, \tilde{W}_x) \end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{O}(|W_x|^2) \end{pmatrix}$$

The first term, since it vanishes in the first component and since $|\tilde{W}_x|$ decays at plus spatial infinity, yields by Young's inequality the estimate

$$\left\langle \begin{pmatrix} 0 \\ g_1(\tilde{W}_x, \tilde{W}_{x_1}) + g_1(\tilde{W}_{x_1}, \tilde{W}_x) \end{pmatrix}, \begin{pmatrix} w_x^I \\ w_x^{II} \end{pmatrix} \right\rangle \leq C \left(\langle (\zeta + |\tilde{W}_{x_1}|) w_x^I, w_x^I \rangle + \|w_x^{II}\|_0^2 \right)$$

which can be treated in the Friedrichs-type estimates. The $(0, \mathcal{O}(|W_x|^2))^T$ nonlinear term may be treated as other source terms in the energy estimates. Specifically, the worst-case term

$$\begin{aligned} & \left\langle \partial_x^k W, \partial_x^k \begin{pmatrix} 0 \\ \mathcal{O}(|W_x|^2) \end{pmatrix} \right\rangle \\ &= -\langle \partial_x^{k+1} w^{II}, \partial_x^{k-1} \mathcal{O}(|W_x|^2) \rangle - \partial_x^k w^{II}(0) \partial_x^{k-1} \mathcal{O}(|W_x|^2)(0) \end{aligned}$$

may be bounded by

$$\|\partial_x^{k+1} w^{II}\|_{L^2} \|W\|_{W^{2,\infty}} \|W\|_{H^k} - \partial_x^k w^{II}(0) \partial_x^{k-1} \mathcal{O}(|W_x|^2)(0).$$

The boundary term will contribute to energy estimates in the form (4.35) of I_b^α , and thus we may use the parabolic equations to get rid of this term as we did in (4.23), (4.24). Thus, we may relax (A3') to (A3), completing the proof of the general case (A1) – (A3) and the proposition. \square

4.2. Proof of nonlinear stability. Defining the perturbation variable $U := \tilde{U} - \bar{U}$, we obtain the nonlinear perturbation equations

$$(4.51) \quad U_t - LU = \sum_j Q^j(U, U_x)_{x_j},$$

where

$$\begin{aligned} (4.52) \quad & Q^j(U, U_x) = \mathcal{O}(|U||U_x| + |U|^2) \\ & Q^j(U, U_x)_{x_j} = \mathcal{O}(|U||U_x| + |U||U_{xx}| + |U_x|^2) \\ & Q^j(U, U_x)_{x_j x_k} = \mathcal{O}(|U||U_{xx}| + |U_x||U_{xx}| + |U_x|^2 + |U||U_{xxx}|) \end{aligned}$$

so long as $|U|$ remains bounded.

For boundary conditions written in U -coordinates, (B) gives

$$(4.53) \quad \begin{aligned} h = \tilde{h} - \bar{h} &= (\tilde{W}(U + \bar{U}) - \tilde{W}(\bar{U}))(0, \tilde{x}, t) \\ &= (\partial \tilde{W} / \partial \tilde{U})(\bar{U}_0)U(0, \tilde{x}, t) + \mathcal{O}(|U(0, \tilde{x}, t)|^2). \end{aligned}$$

in inflow case, where $(\partial \tilde{W} / \partial \tilde{U})(\bar{U}_0)$ is constant and invertible, and

$$(4.54) \quad \begin{aligned} h = \tilde{h} - \bar{h} &= (\tilde{w}^{II}(U + \bar{U}) - \tilde{w}^{II}(\bar{U}))(0, \tilde{x}, t) \\ &= (\partial \tilde{w}^{II} / \partial \tilde{U})(\bar{U}_0)U(0, \tilde{x}, t) + \mathcal{O}(|U(0, \tilde{x}, t)|^2) \\ &= m(\bar{b}_1 \quad \bar{b}_2)(\bar{U}_0)U(0, \tilde{x}, t) + \mathcal{O}(|U(0, \tilde{x}, t)|^2) \\ &= mB(\bar{U}_0)U(0, \tilde{x}, t) + \mathcal{O}(|U(0, \tilde{x}, t)|^2) \end{aligned}$$

for some invertible constant matrix m .

Applying Lemma 3.9 to (4.51), we obtain

$$(4.55) \quad U(x, t) = \mathcal{S}(t)U_0 + \int_0^t \mathcal{S}(t-s) \sum_j \partial_{x_j} Q^j(U, U_x) ds + \Gamma U(0, \tilde{x}, t)$$

where $U(x, 0) = U_0(x)$,

$$(4.56) \quad \Gamma U(0, \tilde{x}, t) := \int_0^t \int_{\mathbb{R}^{d-1}} \left(\sum_j G_{y_j} B^{j1} + G A^1 \right)(x, t-s; 0, \tilde{y}) U(0, \tilde{y}, s) d\tilde{y} ds,$$

and G is the Green function of $\partial_t - L$.

Proof of Theorem 1.7. Define

$$(4.57) \quad \begin{aligned} \zeta(t) &:= \sup_s \left(|U(s)|_{L_x^2} (1+s)^{\frac{d-1}{4}} + |U(s)|_{L_x^\infty} (1+s)^{\frac{d}{2}} \right. \\ &\quad \left. + (|U(s)| + |U_x(s)| + |\partial_{\tilde{x}}^2 U(s)|)_{L_{\tilde{x}, x_1}^{2, \infty}} (1+s)^{\frac{d+1}{4}} \right). \end{aligned}$$

We shall prove here that for all $t \geq 0$ for which a solution exists with $\zeta(t)$ uniformly bounded by some fixed, sufficiently small constant, there holds

$$(4.58) \quad \zeta(t) \leq C(|U_0|_{L^1 \cap H^s} + E_0 + \zeta(t)^2).$$

This bound together with continuity of $\zeta(t)$ implies that

$$(4.59) \quad \zeta(t) \leq 2C(|U_0|_{L^1 \cap H^s} + E_0)$$

for $t \geq 0$, provided that $|U_0|_{L^1 \cap H^s} + E_0 < 1/4C^2$. This would complete the proof of the bounds as claimed in the theorem, and thus give the main theorem.

By standard short-time theory/local well-posedness in H^s , and the standard principle of continuation, there exists a solution $U \in H^s$ on the open time-interval for which $|U|_{H^s}$ remains bounded, and on this interval $\zeta(t)$ is well-defined and continuous. Now, let $[0, T)$ be the maximal interval on which $|U|_{H^s}$ remains strictly bounded by some fixed, sufficiently

small constant $\delta > 0$. By Proposition 4.1, and the Sobolev embedding inequality $|U|_{W^{2,\infty}} \leq C|U|_{H^s}$, we have

$$(4.60) \quad \begin{aligned} |U(t)|_{H^s}^2 &\leq C e^{-\theta t} |U_0|_{H^s}^2 + C \int_0^t e^{-\theta(t-\tau)} \left(|U(\tau)|_{L^2}^2 + |\mathcal{B}_h(\tau)|^2 \right) d\tau \\ &\leq C(|U_0|_{H^s}^2 + E_0^2 + \zeta(t)^2)(1+t)^{-(d-1)/2}. \end{aligned}$$

and so the solution continues so long as ζ remains small, with bound (4.59), yielding existence and the claimed bounds.

Thus, it remains to prove the claim (4.58). First by (4.55), we obtain

$$(4.61) \quad \begin{aligned} |U(t)|_{L^2} &\leq |\mathcal{S}(t)U_0|_{L^2} + \int_0^t |\mathcal{S}_1(t-s)\partial_{x_j}Q^j(s)|_{L^2} ds \\ &\quad + \int_0^t |\mathcal{S}_2(t-s)\partial_{x_j}Q^j(s)|_{L^2} ds + |\Gamma U(0, \tilde{x}, t)|_{L^2} \\ &\leq I_1 + I_2 + I_3 + |\Gamma U(0, \tilde{x}, t)|_{L^2} \end{aligned}$$

where

$$\begin{aligned} I_1 &:= |\mathcal{S}(t)U_0|_{L^2} \leq C(1+t)^{-\frac{d-1}{4}} |U_0|_{L^1 \cap H^3}, \\ I_2 &:= \int_0^t |\mathcal{S}_1(t-s)\partial_{x_j}Q^j(s)|_{L^2} ds \\ &\leq C \int_0^t (1+t-s)^{-\frac{d-1}{4}-\frac{1}{2}} |Q^j(s)|_{L^1} + (1+s)^{-\frac{d-1}{4}} |Q^j(s)|_{L_{\tilde{x},x_1}^{1,\infty}} ds \\ &\leq C \int_0^t (1+t-s)^{-\frac{d-1}{4}-\frac{1}{2}} |U|_{H^1}^2 + (1+t-s)^{-\frac{d-1}{4}} \left(|U|_{L_{\tilde{x},x_1}^{2,\infty}}^2 + |U_x|_{L_{\tilde{x},x_1}^{2,\infty}}^2 \right) ds \\ &\leq C(|U_0|_{H^s}^2 + \zeta(t)^2) \int_0^t \left[(1+t-s)^{-\frac{d-1}{4}-\frac{1}{2}} (1+s)^{-\frac{d-1}{2}} \right. \\ &\quad \left. + (1+t-s)^{-\frac{d-1}{4}} (1+s)^{-\frac{d+1}{2}} \right] ds \\ &\leq C(1+t)^{-\frac{d-1}{4}} (|U_0|_{H^s}^2 + \zeta(t)^2) \end{aligned}$$

and

$$\begin{aligned}
I_3 &:= \int_0^t |\mathcal{S}_2(t-s) \partial_{x_j} Q^j(s)|_{L^2} ds \\
&\leq \int_0^t e^{-\theta(t-s)} |\partial_{x_j} Q^j(s)|_{H^3} ds \\
&\leq C \int_0^t e^{-\theta(t-s)} (|U|_{L^\infty} + |U_x|_{L^\infty}) |U|_{H^5} ds \\
&\leq C \int_0^t e^{-\theta(t-s)} |U|_{H^s}^2 ds \\
&\leq C(|U_0|_{H^s}^2 + \zeta(t)^2) \int_0^t e^{-\theta(t-s)} (1+s)^{-\frac{d-1}{2}} ds \\
&\leq C(1+t)^{-\frac{d-1}{2}} (|U_0|_{H^s}^2 + \zeta(t)^2).
\end{aligned}$$

Meanwhile, for the boundary term $|\Gamma U(0, \tilde{x}, t)|_{L^2}$, we treat two cases separately. First for the inflow case, then by (4.53) we have

$$|U(0, \tilde{x}, t)| \leq C|h(\tilde{x}, t)| + \mathcal{O}(|U(0, \tilde{x}, t)|^2),$$

and thus $|U(0, \tilde{x}, t)| \leq C|h(\tilde{x}, t)|$, provided that $|h|$ is sufficiently small. Therefore under the hypotheses on h in Theorem 1.7, Proposition 3.8 yields

$$|\Gamma U(0, \cdot, \cdot)|_{L_x^2} \leq CE_0(1+t)^{-\frac{d-1}{4}}.$$

Now for the outflow case, recall that in this case $G(x, t; 0, \tilde{y}) \equiv 0$. Thus (4.56) simplifies to

$$(4.62) \quad \Gamma U(0, \tilde{x}, t) = \int_0^t \int_{\mathbb{R}^{d-1}} G_{y_1}(x, t-s; 0, \tilde{y}) B^{11} U(0, \tilde{y}, s) d\tilde{y} ds.$$

To deal with this term, we shall use Proposition 3.8 as in the inflow case. In view of (4.54),

$$|B^{11} U(0, \tilde{y}, s)| \leq C|h(\tilde{y}, s)| + \mathcal{O}(|U(0, \tilde{y}, s)|^2),$$

and assumptions on h are imposed as in Theorem 1.6, so that (3.27) is satisfied. To check the last term $\mathcal{O}(|U(0)|^2)$, using the definition (4.57) of $\zeta(t)$, we have

$$\begin{aligned}
|\mathcal{O}(|U(0, \tilde{y}, s)|^2)|_{L^2} &\leq C|U|_{L^\infty} |U|_{L_{\tilde{x}, x_1}^{2, \infty}} \leq C\zeta^2(t)(1+s)^{-\frac{d}{2}-\frac{d+1}{4}} \\
|\mathcal{O}(|U(0, \tilde{y}, s)|^2)|_{L^\infty} &\leq C|U|_{L^\infty}^2 \leq C\zeta^2(t)(1+s)^{-d}
\end{aligned}$$

and for the term \mathcal{D}_h with h replaced by $\mathcal{O}(|U(0, \tilde{y}, s)|^2)$, using the standard Hölder inequality to get

$$\begin{aligned}
|\mathcal{D}_h|_{L_{\tilde{x}}^1} &\leq C(|U|_{L^{2, \infty}}^2 + |U_x|_{L^{2, \infty}}^2 + |U_{\tilde{x}\tilde{x}}|_{L^{2, \infty}}^2) \leq C\zeta^2(t)(1+s)^{-\frac{d+1}{2}} \\
|\mathcal{D}_h|_{H_{\tilde{x}}^{[(d-1)/2]+5}} &\leq C|U|_{L^\infty} |U|_{H^s} \leq C\zeta^2(t)(1+s)^{-d/2-(d-1)/4}.
\end{aligned}$$

We remark here that Sobolev bounds (4.60) are not good enough for estimates of \mathcal{D}_h in L^1 , requiring a decay at rate $(1+t)^{-d/2-\epsilon}$ for the two-dimensional case (see Proposition 3.8). This is exactly why we have to keep track of $U_{\tilde{x}\tilde{x}}$ in $L^{2, \infty}$ norm in $\zeta(t)$ as well, to gain a bound as above for \mathcal{D}_h .

Therefore applying Proposition 3.8, we also obtain (4.62) for the outflow case. Combining these above estimates yields

$$(4.63) \quad |U(t)|_{L^2} (1+t)^{\frac{d-1}{4}} \leq C(|U_0|_{L^1 \cap H^s} + E_0 + \zeta(t)^2).$$

Next, we estimate

$$(4.64) \quad \begin{aligned} |U(t)|_{L_{\tilde{x}, x_1}^{2, \infty}} &\leq |\mathcal{S}(t)U_0|_{L_{\tilde{x}, x_1}^{2, \infty}} + \int_0^t |\mathcal{S}_1(t-s)\partial_{x_j} Q^j(s)|_{L_{\tilde{x}, x_1}^{2, \infty}} ds \\ &\quad + \int_0^t |\mathcal{S}_2(t-s)\partial_{x_j} Q^j(s)|_{L_{\tilde{x}, x_1}^{2, \infty}} ds + |\Gamma U(0, \tilde{x}, t)|_{L_{\tilde{x}, x_1}^{2, \infty}} \\ &\leq J_1 + J_2 + J_3 + |\Gamma U(0, \tilde{x}, t)|_{L_{\tilde{x}, x_1}^{2, \infty}} \end{aligned}$$

where

$$\begin{aligned} J_1 &:= |\mathcal{S}(t)U_0|_{L_{\tilde{x}, x_1}^{2, \infty}} \leq C(1+t)^{-\frac{d+1}{4}} |U_0|_{L^1 \cap H^4} \\ J_2 &:= \int_0^t |\mathcal{S}_1(t-s)\partial_{x_j} Q^j(s)|_{L_{\tilde{x}, x_1}^{2, \infty}} ds \\ &\leq C \int_0^t (1+t-s)^{-\frac{d+1}{4}-\frac{1}{2}} |Q^j(s)|_{L^1} + (1+s)^{-\frac{d+1}{4}} |Q^j(s)|_{L_{\tilde{x}, x_1}^{1, \infty}} ds \\ &\leq C \int_0^t (1+t-s)^{-\frac{d+1}{4}-\frac{1}{2}} |U|_{H^1}^2 + (1+t-s)^{-\frac{d+1}{4}} \left(|U|_{L_{\tilde{x}, x_1}^{2, \infty}}^2 + |U_x|_{L_{\tilde{x}, x_1}^{2, \infty}}^2 \right) ds \\ &\leq C(|U_0|_{H^s}^2 + \zeta(t)^2) \int_0^t (1+t-s)^{-\frac{d+1}{4}-\frac{1}{2}} (1+s)^{-\frac{d-1}{2}} \\ &\quad + (1+t-s)^{-\frac{d+1}{4}} (1+s)^{-\frac{d+1}{2}} ds \\ &\leq C(1+t)^{-\frac{d+1}{4}} (|U_0|_{H^s}^2 + \zeta(t)^2) \end{aligned}$$

and (by Moser's inequality)

$$\begin{aligned} J_3 &:= \int_0^t |\mathcal{S}_2(t-s)\partial_{x_j} Q^j(s)|_{L_{\tilde{x}, x_1}^{2, \infty}} ds \\ &\leq C \int_0^t e^{-\theta(t-s)} |\partial_{x_j} Q^j(s)|_{H^4} ds \\ &\leq C \int_0^t e^{-\theta(t-s)} |U|_{L_x^\infty} |U|_{H^6} ds \\ &\leq C(|U_0|_{H^s}^2 + \zeta(t)^2) \int_0^t e^{-\theta(t-s)} (1+s)^{-\frac{d}{2}} (1+s)^{-\frac{d-1}{4}} ds \\ &\leq C(1+t)^{-\frac{d+1}{4}} (|U_0|_{H^s}^2 + \zeta(t)^2). \end{aligned}$$

These estimates together with similar treatment for the boundary term yield

$$(4.65) \quad |U(t)|_{L_{\tilde{x}, x_1}^{2, \infty}} (1+t)^{\frac{d+1}{4}} \leq C(|U_0|_{L^1 \cap H^s} + E_0 + \zeta(t)^2).$$

Similarly, we have the same estimate for $|U_x(t)|_{L_{\tilde{x},x_1}^{2,\infty}}$. Indeed, we have

$$\begin{aligned}
 (4.66) \quad |U_x(t)|_{L_{\tilde{x},x_1}^{2,\infty}} &\leq |\partial_x \mathcal{S}(t)U_0|_{L_{\tilde{x},x_1}^{2,\infty}} + \int_0^t |\partial_x \mathcal{S}_1(t-s)\partial_{x_j} Q^j(s)|_{L_{\tilde{x},x_1}^{2,\infty}} ds \\
 &\quad + \int_0^t |\partial_x \mathcal{S}_2(t-s)\partial_{x_j} Q^j(s)|_{L_{\tilde{x},x_1}^{2,\infty}} ds + |\partial_x \Gamma U(0, \tilde{x}, t)|_{L_{\tilde{x},x_1}^{2,\infty}} \\
 &\leq K_1 + K_2 + K_3 + |\partial_x \Gamma U(0, \tilde{x}, t)|_{L_{\tilde{x},x_1}^{2,\infty}}
 \end{aligned}$$

where K_2 and K_3 are treated exactly in the same way as the treatment of J_2, J_3 , yet in the first term of K_2 it is a bit better by a factor $t^{-1/2}$. Similar bounds hold for $|U_{\tilde{x}\tilde{x}}|$ in $L^{2,\infty}$, noting that there are no higher derivatives in x_1 involved and thus similar to those in (4.64).

Finally, we estimate the L^∞ norm of U . By Duhamel's formula (4.55), we obtain

$$\begin{aligned}
 (4.67) \quad |U(t)|_{L^\infty} &\leq |\mathcal{S}(t)U_0|_{L^\infty} + \int_0^t |\mathcal{S}_1(t-s)\partial_{x_j} Q^j(s)|_{L^\infty} ds \\
 &\quad + \int_0^t |\mathcal{S}_2(t-s)\partial_{x_j} Q^j(s)|_{L^\infty} ds + |\Gamma U(0, \tilde{x}, t)|_{L^\infty} \\
 &\leq L_1 + L_2 + L_3 + |\Gamma U(0, \tilde{x}, t)|_{L^\infty}
 \end{aligned}$$

where the boundary term is treated in the same way as above, and for $|\gamma| = [(d-1)/2] + 2$,

$$\begin{aligned}
 L_1 &:= |\mathcal{S}(t)U_0|_{L^\infty} \leq C(1+t)^{-\frac{d}{2}} |U_0|_{L^1 \cap H^{|\gamma|+3}}, \\
 L_2 &:= \int_0^t |\mathcal{S}_1(t-s)\partial_{x_j} Q^j(s)|_{L^\infty} ds \\
 &\leq C \int_0^t (1+t-s)^{-\frac{d}{2}-\frac{1}{2}} |Q^j(s)|_{L^1} + (1+s)^{-\frac{d}{2}} |Q^j(s)|_{L_{\tilde{x},x_1}^{1,\infty}} ds \\
 &\leq C \int_0^t (1+t-s)^{-\frac{d}{2}-\frac{1}{2}} |U|_{H^1}^2 + (1+t-s)^{-\frac{d}{2}} \left(|U|_{L_{\tilde{x},x_1}^{2,\infty}}^2 + |U_x|_{L_{\tilde{x},x_1}^{2,\infty}}^2 \right) ds \\
 &\leq C(|U_0|_{H^s}^2 + \zeta(t)^2) \int_0^t \left[(1+t-s)^{-\frac{d}{2}-\frac{1}{2}} (1+s)^{-\frac{d-1}{2}} \right. \\
 &\quad \left. + (1+t-s)^{-\frac{d}{2}} (1+s)^{-\frac{d+1}{2}} \right] ds \\
 &\leq C(1+t)^{-\frac{d}{2}} (|U_0|_{H^s}^2 + \zeta(t)^2)
 \end{aligned}$$

and (again by Moser's inequality),

$$\begin{aligned}
L_3 &:= \int_0^t |\mathcal{S}_2(t-s) \partial_{x_j} Q^j(s)|_{L^\infty} ds \\
&\leq \int_0^t |\mathcal{S}_2(t-s) \partial_{x_j} Q^j(s)|_{H^{|\gamma|}} ds \\
&\leq \int_0^t e^{-\theta(t-s)} |\partial_x Q^j(s)|_{H^{|\gamma|+3}} ds \\
&\leq C \int_0^t e^{-\theta(t-s)} |U|_{L^\infty} |U|_{H^{|\gamma|+5}} ds \\
&\leq C(|U_0|_{H^s}^2 + \zeta(t)^2) \int_0^t e^{-\theta(t-s)} (1+s)^{-\frac{d}{2}} (1+s)^{-\frac{d-1}{4}} ds \\
&\leq C(1+t)^{-\frac{d}{2}} (|U_0|_{H^s}^2 + \zeta(t)^2).
\end{aligned}$$

Therefore we have obtained

$$(4.68) \quad |U(t)|_{L_x^\infty} (1+t)^{\frac{d}{2}} \leq C(|U_0|_{L^1 \cap H^s} + E_0 + \zeta(t)^2)$$

and thus completed the proof of claim (4.58), and the main theorem. \square

APPENDIX A. PHYSICAL DISCUSSION IN THE ISENTROPIC CASE

In this appendix, we revisit in slightly more detail the drag-reduction problem sketched in Examples 1.1–1.2, in the simplified context of the two-dimensional isentropic case. Following the notation of [GMWZ5], consider the two-dimensional isentropic compressible Navier–Stokes equations

$$(A.1) \quad \rho_t + (\rho u)_x + (\rho v)_y = 0,$$

$$(A.2) \quad (\rho u)_t + (\rho u^2)_x + (\rho uv)_y + p_x = (2\mu + \eta)u_{xx} + \mu u_{yy} + (\mu + \eta)v_{xy},$$

$$(A.3) \quad (\rho v)_t + (\rho uv)_x + (\rho v^2)_y + p_y = \mu v_{xx} + (2\mu + \eta)v_{yy} + (\mu + \eta)u_{yx}$$

on the half-space $y > 0$, where ρ is density, u and v are velocities in x and y directions, and $p = p(\rho)$ is pressure, and $\mu > |\eta| \geq 0$ are coefficients of first (“dynamic”) and second viscosity, making the standard monotone pressure assumption $p'(\rho) > 0$.

We imagine a porous airfoil lying along the x -axis, with constant imposed normal velocity $v(0) = V$ and zero transverse relative velocity $u(0) = 0$ imposed at the airfoil surface, and seek a laminar boundary-layer flow $(\rho, u, v)(y)$ with transverse relative velocity u_∞ a short distance away the airfoil, with $|V|$ much less than the sound speed c_∞ and $|u_\infty|$ of an order roughly comparable to c_∞ .

A.1. Existence. The possible boundary-layer solutions have been completely categorized in this case in Section 5.1 of [GMWZ5]. We here cite the relevant conclusions, referring to [GMWZ5] for the (straightforward) justifying computations.

A.1.1. *Outflow case* ($V < 0$). In the outflow case, the scenario described above corresponds to case (5.15) of [GMWZ5], in which it is found that the only solutions are purely *transverse* flows

$$(A.4) \quad (\rho, v) \equiv (\rho_0, V), \quad u(y) = u_\infty(1 - e^{\rho_0 V y / \mu}),$$

varying only in the transverse velocity u . The drag force per unit length at the airfoil, by Newton's law of viscosity, is

$$(A.5) \quad \mu \bar{u}_y|_{y=0} = u_\infty \rho_\infty |V|,$$

since momentum $m := \rho_0 V = \rho_\infty V$ is constant throughout the layer, so that (ρ_∞, u_∞) being imposed by ambient conditions away from the wing) *drag is proportional to the speed $|V|$ of the imposed normal velocity.*

A.1.2. *Inflow case* ($V > 0$). Consulting again [GMWZ5] (p. 61), we find for $V > 0$ with specified $(\rho, u, v)(0)$ of the orders described above, the only solutions are purely *normal* flows,

$$(A.6) \quad u \equiv u(0), \quad (\rho, v) = (\rho, v)(y),$$

varying only in the normal velocity v . Thus, it is not possible to reconcile the velocity $u(0)$ at the airfoil with the velocity $u_\infty \gg c$ some distance away.

As discussed in [MN], the expected behavior in such a case consists rather of a combination of a boundary-layer at $y = 0$ and one or more elementary planar shock, rarefaction, or contact waves moving away from $y = 0$: in this case a shear wave moving with normal fluid velocity V into the half-space, across which the transverse velocity changes from zero to u_∞ . That is, a characteristic layer analogous to the solid-boundary case *detaches* from the airfoil and travels outward into the flow field. In this case, one would not expect drag reduction compared to the solid-boundary case, but rather some increase.

A.2. Stability. If we consider one-dimensional stability, or stability with respect to perturbations depending only on y , we find that the linearized eigenvalue equations decouple into the constant-coefficient linearized eigenvalue equations for (ρ, v) about a constant layer $(\rho, v) \equiv (\rho_0, V)$, and the scalar linearized eigenvalue equation

$$(A.7) \quad \lambda \bar{\rho} u + m u_y = \mu u_{yy}$$

associated with the constant-coefficient convection-diffusion equation $\bar{\rho} u_t + m u = \mu u_{yy}$, $m := \bar{\rho} \bar{v} \equiv \rho_0 V$, $\bar{\rho} \equiv \rho_0$. As the constant layer (ρ_0, V) is stable by Corollary 1.5 or direct calculation (Fourier transform), and (A.7) is stable by direct calculation, we may thus conclude that purely transverse layers are *one-dimensionally stable*.

Considered with respect to general perturbations, the equations do not decouple, nor do they reduce to constant-coefficient form, but to a second order system whose coefficients are quadratic polynomials in $e^{\rho_0 V y}$. It would be very interesting to try to resolve the question of spectral stability by direct solution using this special form, or, alternatively, to perform a numerical study as done in [HLyZ2] for the multi-dimensional shock wave case.

Remark A.1. For general laminar boundary layers $(\bar{\rho}, \bar{u}, \bar{v})(y)$, the one-dimensional stability problem, now variable-coefficient, does not completely decouple, but has triangular

form, breaking into a system in (ρ, v) alone and an equation in u forced by (ρ, v) . Stability with respect to general perturbations, therefore, is equivalent to stability with respect to perturbations of form $(\rho, 0, v)$ or $(0, u, 0)$. For perturbations $(\rho, u, v) = (0, u, 0)$, the u equation again becomes (A.7), with μ, m still constant, but $\bar{\rho}$ varying in y . Taking the real part of the complex L^2 inner product of u against (A.7) gives

$$\Re \lambda \|u\|_{L^2}^2 + \|u_y\|_{L^2}^2 = 0,$$

hence for $\Re \lambda \geq 0$, $u \equiv \text{constant} = 0$. Thus, the layer is one-dimensionally stable if and only if the normal part $(\bar{\rho}, \bar{v})$ is stable with respect to perturbations (ρ, v) . Stability of normal layers was studied in [CHNZ] for a γ -law gas $p(\rho) = a\rho^\gamma$, $1 \leq \gamma \leq 3$, with the conclusion that *all layers are one-dimensionally stable*, independent of amplitude, in the general inflow and compressive outflow cases. Hence, we can make the same conclusion for full layers $(\bar{\rho}, \bar{u}, \bar{v})$. In the present context, this includes all cases except for suction with supersonic velocity $|V| > c_\infty$, which in the notation of [CHNZ] is of *expansive outflow* type (expected also to be stable, but not considered in [CHNZ]), since $|\bar{v}|$ is decreasing with y , so that density $\bar{\rho}$ (since $m = \bar{\rho}\bar{v} \equiv \text{constant}$) is increasing.

A.3. Discussion. Note that we do not achieve by subsonic boundary suction an exact laminar flow connecting the values $(u, v) = (0, V)$ at the wing to the values $(u_\infty, 0)$ of the ambient flow at infinity, but rather to an intermediate value (u_∞, V) . That is, we trade a large variation u_∞ in shear for a possibly small variation V in normal velocity, which appears now as a boundary condition for the outer, approximately Euler flow away from the boundary layer. Whether the full solution is stable appears to be a question concerning also nonstationary Euler flow. It is not clear either what is the optimal outflux velocity V . From (A.5) and the discussion just above, it appears desirable to minimize $|V|$, since this minimizes both drag and the imbalance between flow v_∞ just outside the boundary layer and the ambient flow at infinity. On the other hand, we expect that stability becomes more delicate in the characteristic limit $V \rightarrow 0^-$, in the sense that the size of the basin of attraction of the boundary layer shrinks to zero (recall, we have ignored throughout our analysis the size of the basin of attraction, taking perturbations as small as needed without keeping track of constants). These would be quite interesting issues for further investigation.

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